

OBSTRUCTION SETS AND EXTENSIONS OF GROUPS

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ABSTRACT. Let X be a nice variety over a number field k . We characterise in pure “descent-type” terms some inequivalent obstruction sets refining the inclusion $X(\mathbb{A}_k)^{\text{ét,Br}} \subset X(\mathbb{A}_k)^{\text{Br}_1}$. In the first part, we apply ideas from the proof of $X(\mathbb{A}_k)^{\text{ét,Br}} = X(\mathbb{A}_k)^{\mathcal{L}_k}$ by Skorobogatov and Demarche to new cases, by proving a comparison theorem for obstruction sets. In the second part, we show that if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ are such that $\mathcal{B} \subset \text{Ext}(\mathcal{A}, \mathcal{U}_k)$, then $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$. This allow us to conclude, among other things, that $X(\mathbb{A}_k)^{\text{ét,Br}} = X(\mathbb{A}_k)^{\mathcal{R}_k}$ and that $X(\mathbb{A}_k)^{\text{Sol,Br}_1} = X(\mathbb{A}_k)^{\text{Sol}_k}$.

1. INTRODUCTION

1.1. General notation and background. In this paper, k will always be a number field and \bar{k} a fixed algebraic closure of k . Let Ω_k be the set of places of k . If $v \in \Omega_k$, we denote by k_v the completion of k at v . By a **variety X over k** , we will mean a reduced, separated scheme of finite type over k . We say that a variety is nice if it is smooth, projective, and geometrically integral. If l/k is any field extension, we write X_l for $X \times_{\text{Spec}(k)} \text{Spec}(l)$; when $l = \bar{k}$, we simply write \bar{X} for $X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$. Let X be a variety over k . If l is any field containing k , we denote by

$$X(l) := \text{Hom}_{\text{Spec}(k)}(\text{Spec}(l), X)$$

the set of l -rational points of X . There is a diagonal embedding $X(k) \subset X(\mathbb{A}_k)$, where $X(\mathbb{A}_k)$ is the set of adelic points of X and is endowed with the restricted product topology. We remark that if X is proper (e.g. if X is projective), then $X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v)$.

A **linear algebraic group over k** is an affine variety over k that has an algebraic group law structure (also defined over k). The reader interested in the theory of linear algebraic groups is referred to [Bor91], [Hum81], and [Spr98], or, for a more modern approach, to [Mil15]. Given a linear algebraic group G over k , a k -scheme Y with a G -action $\mu : Y \times G \rightarrow Y$, and a G -morphism $f : Y \rightarrow X$, we say that Y is a (right) G -torsor over X if f is faithfully flat and if the morphism $Y \times G \rightarrow Y \times_X Y$ given by $(y, g) \mapsto (y, \mu(y, g))$ is an isomorphism.

Let $\mathcal{L}_k := \{G : G \text{ is a linear algebraic group over } k\} / \sim$, where $G_1 \sim G_2$ if and only if G_1 is k -isomorphic to G_2 as algebraic k -groups. If $\mathcal{S} \subset \mathcal{L}_k$, we write “ $(f, Y, G) \in \mathcal{S}(X)$ ” to mean “ $f : Y \rightarrow X$ is a G -torsor over X , with $G \in \mathcal{S}$ ”; when the emphasis on f is not needed, we just write “ (Y, G) ” instead of “ (f, Y, G) ”. The pointed set $H^1(X, G) := \check{H}_{\text{fppf}}^1(X, G)$, which is a group whenever G is abelian, classifies G -torsors over X up to k -isomorphism ([Sko01], §2.2); we usually denote by $[Y]$ the k -isomorphism class in $H^1(X, G)$ of the G -torsor $f : Y \rightarrow X$.

Given a G -torsor $f : Y \rightarrow X$ over X and a 1-cocycle $\tau \in Z^1(k, G)$, we can twist Y by τ : we can construct a G^τ -torsor $f^\tau : Y^\tau \rightarrow X$ over X , where G^τ and Y^τ are “twisted” versions of G and Y (see Chapter 2 of [Sko01] for more details); we remark that, up to k -isomorphism, this

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construction only depends on the class $[\tau] \in H^1(k, G)$. For more on the theory of torsors, we refer the reader to Chapter 2 of [Sko01].

For any $(f, Y, G) \in \mathcal{S}(X)$, we write $X(\mathbb{A}_k)^f := \bigcup_{[\tau] \in H^1(k, G)} f^\tau(Y^\tau(\mathbb{A}_k))$, and for any $\mathcal{S} \subset \mathcal{L}_k$, we write

$$X(\mathbb{A}_k)^\mathcal{S} := \bigcap_{G \in \mathcal{S}} \bigcap_{[Y] \in H^1(X, G)} X(\mathbb{A}_k)^f.$$

The Brauer group functor $\text{Br} : \mathbf{Sch}_k^{\text{op}} \rightarrow \mathbf{Ab}$ is given by $Y \mapsto \check{H}_{\text{ét}}^2(Y, \mathbb{G}_m)$, and the Brauer group of X is simply $\text{Br}(X)$. The algebraic Brauer group of X is given by $\text{Br}_1(X) := \ker(\text{Br}(X) \rightarrow \text{Br}(\overline{X}))$, and one can check that $\text{Br}_1 : \mathbf{Sch}_k^{\text{op}} \rightarrow \mathbf{Ab}$ is a subfunctor of Br . Recall that the Brauer-Manin pairing $\langle -, - \rangle_{BM} : X(\mathbb{A}_k) \times \text{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$ is defined by $((x_v), \alpha) \mapsto \sum_{v \in \Omega_k} \text{inv}_v \alpha(x_v)$, where $\text{inv}_v : \text{Br} k_v \rightarrow \mathbb{Q}/\mathbb{Z}$, for $v \in \Omega_k$, is the invariant map coming from local class field theory. The Brauer-Manin set is then given by

$$X(\mathbb{A}_k)^{\text{Br}} := \bigcap_{\alpha \in \text{Br}(X)} \{(x_v) \in X(\mathbb{A}_k) : \langle (x_v), \alpha \rangle_{BM} = 0\}.$$

We define the algebraic Brauer-Manin set $X(\mathbb{A}_k)^{\text{Br}_1}$ similarly, by taking the intersection over $\alpha \in \text{Br}_1(X)$ instead. We have that $\overline{X(k)} \subset X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)^{\text{Br}_1} \subset X(\mathbb{A}_k)$, where the first inclusion follows from global class field theory and the continuity of $\langle -, \alpha \rangle_{BM} : X(\mathbb{A}_k) \rightarrow \mathbb{Q}/\mathbb{Z}$ for each $\alpha \in \text{Br} X$.

More generally, let $\mathcal{O} : \mathbf{Sch}_k^{\text{op}} \rightarrow \mathbf{Sets}$ be any functor from the opposite category of schemes over k to the category of sets. Let $W \in \mathbf{Sch}_k$. For each $\alpha \in \mathcal{O}(W)$, there is a commutative diagram

$$\begin{array}{ccc} W(k) & \longrightarrow & W(\mathbb{A}_k) \\ \text{ev}_\alpha \downarrow & & \downarrow \text{ev}_\alpha \\ \mathcal{O}(k) & \longrightarrow & \mathcal{O}(\mathbb{A}_k), \end{array}$$

where the horizontal maps are the diagonal maps and where the vertical maps are the evaluation maps (cf. [Poo, §8.1] for more details). We set

$$W(\mathbb{A}_k)^\mathcal{O} := \bigcap_{\alpha \in \mathcal{O}(W)} \{(w_v) \in W(\mathbb{A}_k) : \text{ev}_\alpha((w_v)) \in \text{Im}(\mathcal{O}(k) \rightarrow \mathcal{O}(\mathbb{A}_k))\}.$$

Hence, for any subset $\mathcal{S} \subset \mathcal{L}_k$, we can define

$$(1.1) \quad X(\mathbb{A}_k)^{\mathcal{S}, \mathcal{O}} := \bigcap_{G \in \mathcal{S}} \bigcap_{[Y] \in H^1(X, G)} \bigcup_{[\tau] \in H^1(k, G)} f^\tau(Y^\tau(\mathbb{A}_k)^\mathcal{O}).$$

Let $\mathcal{F}_k \subset \mathcal{L}_k$ be the set of finite algebraic groups over k (up to k -isomorphisms). Following the refinements studied in [Sto07], we consider the chain of inclusions $\mathcal{F}_k^{\text{Ab}} \subset \mathcal{F}_k^{\text{Sol}} \subset \mathcal{F}_k$, where $\mathcal{F}_k^{\text{Ab}}$ and $\mathcal{F}_k^{\text{Sol}}$ are, respectively, the subsets of commutative and solvable elements of \mathcal{F}_k . By taking $\mathcal{S} = \mathcal{F}_k$ and $\mathcal{O} = \text{Br}$ or $\mathcal{O} = \text{Br}_1$ in (1.1), we get, respectively, the étale-Brauer set, usually denoted by $X(\mathbb{A}_k)^{\text{ét}, \text{Br}}$, and the algebraic étale-Brauer set, denoted by $X(\mathbb{A}_k)^{\text{ét}, \text{Br}_1}$. Similarly, by taking $\mathcal{S} = \mathcal{F}_k^{\text{Sol}}$ or $\mathcal{S} = \mathcal{F}_k^{\text{Ab}}$ and $\mathcal{O} = \text{Br}$, we obtain sets that we will denote by $X(\mathbb{A}_k)^{\text{Sol}, \text{Br}}$ and $X(\mathbb{A}_k)^{\text{Ab}, \text{Br}}$, respectively; by taking $\mathcal{S} = \mathcal{F}_k^{\text{Sol}}$ or $\mathcal{S} = \mathcal{F}_k^{\text{Ab}}$ and $\mathcal{O} = \text{Br}_1$, we obtain sets that we will denote by $X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1}$ and $X(\mathbb{A}_k)^{\text{Ab}, \text{Br}_1}$, respectively.

1.2. Motivation. Let k be a number field. In general, a family $\{X_\omega\}$ of nice varieties over k can satisfy interesting arithmetic properties depending on the interplay between the set of adelic points $X(\mathbb{A}_k)$ and the set of rational points $X(k)$, for all $X \in \{X_\omega\}$. For example, we say that $\{X_\omega\}$ satisfies the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ implies that $X(k) \neq \emptyset$, for all $X \in \{X_\omega\}$; we remark that the opposite implication is clear, as we always have $X(k) \subset X(\mathbb{A}_k)$. Another example, if one is more interested in density properties, is the following: we say that $\{X_\omega\}$ satisfies weak approximation if $X(k) \neq \emptyset$ and $\overline{X(k)} = X(\mathbb{A}_k)$, for all $X \in \{X_\omega\}$.

Often, however, some of these arithmetic properties fail to hold, since $X(\mathbb{A}_k)$ is, in some sense, too big to detect whichever feature we are looking for. (To give an idea, it is common to have $X(\mathbb{A}_k) \neq \emptyset$ but $X(k) = \emptyset$ – a clear failure of the Hasse principle).

When this happens, we can try to refine $X(\mathbb{A}_k)$ by cutting out **obstruction sets** $X(\mathbb{A}_k)^\mathfrak{D} \subset X(\mathbb{A}_k)$ in a suitable way, where “suitable” depends on the context. For example, if we are considering the Hasse principle, we need the inclusion $X(k) \subset X(\mathbb{A}_k)^\mathfrak{D}$ to hold; we then say that there is an \mathfrak{D} -obstruction to the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^\mathfrak{D} = \emptyset$, and that the \mathfrak{D} -principle holds if $X(\mathbb{A}_k)^\mathfrak{D} \neq \emptyset$ implies that $X(k) \neq \emptyset$ (i.e. if the Hasse principle holds with “ $X(\mathbb{A}_k)$ ” replaced by “ $X(\mathbb{A}_k)^\mathfrak{D}$ ”). Similarly, if we are interested in weak approximation, we want $\overline{X(k)} \subset X(\mathbb{A}_k)^\mathfrak{D}$; we then say that there is an \mathfrak{D} -obstruction to weak approximation if $X(\mathbb{A}_k)^\mathfrak{D} \neq X(\mathbb{A}_k)$, and that the \mathfrak{D} -weak approximation holds if $X(k) \neq \emptyset$ and $\overline{X(k)} = X(\mathbb{A}_k)^\mathfrak{D}$ (assuming $X(\mathbb{A}_k)^\mathfrak{D} = \overline{X(\mathbb{A}_k)^\mathfrak{D}}$).

Obstruction sets are thus useful objects to study, as they help us measuring how far varieties are from satisfying interesting arithmetic properties concerning rational points. The general study of these objects took off in the 1970’s with the work of Manin ([Man71]), and is still a very active area of research in arithmetic geometry – see, for example, [Har96], [Sko99], [Poo10], [HS13] for some more recent developments.

There are two different, somewhat competing, approaches in defining obstruction sets: we can either make use of the Brauer group, as is the case for e.g. $X(\mathbb{A}_k)^{\text{ét,Br}}$, or use the more classical “pure” descent on torsors under linear algebraic groups (as is the case for e.g. $X(\mathbb{A}_k)^{\mathcal{L}_k}$), as introduced by Colliot-Thélène and Sansuc in [CTS87]. An interesting task, then, is to try to reconcile these different approaches, that is, to provide a “translation” between obstruction sets defined in a “Brauer-type” language and obstruction sets defined in a “descent-type” language. There are some important results in the literature in this direction, whenever X is a nice variety over a number field k : Skorobogatov has shown that $X(\mathbb{A}_k)^{\text{Br}_1} = X(\mathbb{A}_k)^{\mathcal{M}_k}$, where $\mathcal{M}_k \subset \mathcal{L}_k$ is the set of (k -isomorphism classes of) linear algebraic groups of multiplicative type over k (see [Sko99], Theorem 3; a less general result, requiring $\text{Pic}\overline{X}$ to be torsion-free, had been proven by Colliot-Thélène and Sansuc in [CTS87]); a result by Harari (cf. [Har02], Théorème 2 (ii)), together with a result of Gabber (cf. [dJ]) and Proposition 5.3.4 in [Sko01], implies that $X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)^{\mathcal{C}_k}$, where $\mathcal{C}_k \subset \mathcal{L}_k$ is the set of (k -isomorphism classes of) connected linear algebraic groups over k ; the articles by Demarche [Dem09] and Skorobogatov [Sko09] show that $X(\mathbb{A}_k)^{\text{ét,Br}} = X(\mathbb{A}_k)^{\mathcal{L}_k}$.

Now, any linear algebraic group can be decomposed into simpler building blocks: finite étale algebraic groups, connected linear algebraic groups, reductive linear algebraic groups, solvable linear algebraic groups, semisimple linear algebraic groups, linear algebraic tori, and unipotent linear algebraic groups. A natural question, then, is:

Question 1.1. If $\mathcal{S} \subset \mathcal{L}_k$ is the set of any of the above building blocks, what is (if it exists) the translation of $X(\mathbb{A}_k)^{\mathcal{S}}$ in “Brauer-type” terms?

Symmetrically, one can also ask:

Question 1.2. Given any natural enough Brauer-type obstruction set (such as, for example, $X(\mathbb{A}_k)^{\text{Br}}$, $X(\mathbb{A}_k)^{\text{ét,Br}}$, $X(\mathbb{A}_k)^{\text{Sol,Br}_1}$, and so on), what is (if it exists) its translation in pure “descent-type” terms?

The above mentioned results answer these questions for all linear algebraic groups of multiplicative type (symmetrically, for the algebraic Brauer-Manin set), for all connected linear algebraic groups (symmetrically, for the Brauer-Manin set), and for all linear algebraic groups (symmetrically, for the étale-Brauer set). The aim of this article is to answer Questions 1.1 and 1.2 in some other cases: for unipotent, reductive, and solvable linear algebraic groups, and for the Brauer-type obstruction sets $X(\mathbb{A}_k)^{\text{Sol,Br}}$, $X(\mathbb{A}_k)^{\text{ét,Br}_1}$, and $X(\mathbb{A}_k)^{\text{Sol,Br}_1}$.

1.3. Main results and structure of the paper. In §2, we give some properties of linear algebraic groups. In §§3, 4, we follow closely techniques and ideas from [Dem09], [Sko09], and [Sto07], to prove, under certain conditions, a comparison theorem for obstruction sets.

Theorem (Theorem 3.1). *Let X be a nice variety over k . Let $\mathcal{O} : \mathbf{Sch}_k^{\text{opp}} \rightarrow \mathbf{Sets}$ be any functor such that $X(\mathbb{A}_k)^{\mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Br}_1}$, and let $\mathcal{S}_k \subset \mathcal{L}_k$ be subject to some conditions. Then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)},$$

where $\text{Ext}(\mathcal{F}_k, \mathcal{S}_k) \subset \mathcal{L}_k$ is the subset of linear algebraic groups over k that can be written as extensions of elements in \mathcal{F}_k by elements in \mathcal{S}_k (up to k -isomorphisms). The same result holds if we replace “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ”.

In §5, by applying Theorem 3.1, we are able to translate the natural “Brauer-type” obstruction sets $X(\mathbb{A}_k)^{\text{Sol,Br}}$, $X(\mathbb{A}_k)^{\text{ét,Br}_1}$, and $X(\mathbb{A}_k)^{\text{Sol,Br}_1}$ in a pure “descent-type” language – see, respectively, Theorem 5.1, Corollary 5.10, and Corollary 5.10.

Let $\mathcal{U}_k \subset \mathcal{L}_k$ be the set of (k -isomorphism classes of) unipotent linear algebraic groups over k . In §6, we prove the following.

Theorem (Theorem 6.1). *Let X be a smooth, proper, geometrically integral variety over k . Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \text{Ext}(\mathcal{A}, \mathcal{U}_k)$. (Here, $\text{Ext}(\mathcal{A}, \mathcal{U}_k) \subset \mathcal{L}_k$ is the set of (k -isomorphism classes of) linear algebraic groups over k that can be written as extensions of elements in \mathcal{A} by elements in \mathcal{U}_k .) Then $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$.*

From Theorem 6.1, we then get a series of corollaries (cf. Corollaries 6.8, 6.10, 6.12, 6.14, 6.15, 6.17). In particular,

- (1) $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)$ (Corollary 6.8);
- (2) $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$ (Corollary 6.10). Here, $\mathcal{R}_k \subset \mathcal{L}_k$ is the set of (k -isomorphism classes of) reductive linear algebraic groups over k ;
- (3) $X(\mathbb{A}_k)^{\text{Sol,Br}_1} = X(\mathbb{A}_k)^{\text{Sol}_k}$ (Corollary 6.14). Here, $\text{Sol}_k \subset \mathcal{L}_k$ is the set of (k -isomorphism classes of) of solvable linear algebraic groups over k .

Finally, in §7, we summarise the relations between the different obstruction sets considered in this paper.

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2. EXTENSIONS OF LINEAR ALGEBRAIC GROUPS

Linear algebraic groups play an essential role in this paper, so we start by recalling some of their basic properties. Although some of these properties hold for more general fields, throughout this section we will just consider linear algebraic groups over fields of characteristic 0; this will spare us of many technical complications (not needed in this paper anyway).

Let K be a field of characteristic 0.

Definition 2.1. An (algebraic) K -group is a group object in the category \mathbf{AlgVar}_K of algebraic varieties over K . A linear algebraic K -group is a group object in the category \mathbf{AffVar}_K of affine varieties over K . If G_1, G_2 are algebraic K -groups, a map $\phi : G_1 \rightarrow G_2$ is morphism of algebraic K -groups if it is a K -morphism of K -varieties which is also a homomorphism of groups.

Remark 2.2. In characteristic 0, any algebraic group is always smooth (see [Car62]). Smooth affine algebraic groups are the same as closed algebraic subgroups of GL_n , for some n – hence the name “linear algebraic groups”.

Example 2.3. The multiplicative group $\mathbb{G}_{m,K} = (\mathrm{Spec}(K[x, x^{-1}]), \cdot)$ and the additive group $\mathbb{G}_{a,K} = (\mathrm{Spec}(K[x]), +)$ are linear algebraic K -groups that can be represented, respectively, as

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in K^\times \right\} \subset \mathrm{GL}_2(K) \text{ and } \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in K \right\} \subset \mathrm{GL}_2(K).$$

2.1. Closure properties of linear algebraic groups. For convenience, we remind the reader of some definitions and, without proofs, of some “closure” properties of algebraic groups over K .

- (C1) Let G be a K -group. By a (closed) K -subgroup H of G , written $H \leq_K G$, we mean a K -group H (with multiplication and identity induced from those of G) that is also a K -closed subvariety of G (with respect to the Zariski topology).
- (C2) Let $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ be a short exact sequence of algebraic K -groups. Then B is affine if and only if A and C are affine ([DGA], VI.B, 9.2(viii)). Hence, since we are in characteristic 0, B is linear if and only if A and C are linear.
- (C3) If G is an algebraic K -group and H is a normal (closed) K -subgroup of G , then G/H exists and is an algebraic K -group ([DGA], VI.A, Théorèmes 3.2 and 5.2), which is linear if G is linear.
- (C4) Let G be a K -group. We denote by G^0 the connected component of the identity of G , i.e. the unique connected component of G that contains the identity. This is a (closed) normal K -subgroup of G of finite index, whose cosets are the connected (equivalently, irreducible) components of G (see [Bor91], (1.2)). Note that if G is a linear algebraic K -group, then so is G^0 .
- (C5) Let G be a K -group. Then the radical $R(G)$ of G is the largest connected solvable normal (closed) K -subgroup of G ; the set $R_u(G)$ of all unipotent elements of $R(G)$

is the unipotent radical of G , which is the largest connected unipotent normal closed K -subgroup of G . Note that $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$.

- (C6) Let G and G' be K -groups and let $\phi : G \rightarrow G'$ be a morphism of K -groups. Then $\phi(G)$ is a (closed) normal K -subgroup of G' ([Spr98], Proposition 2.2.5) and, similarly, $\ker(\phi)$ is a (closed) K -subgroup of G .
- (C7) If G is a connected linear algebraic K -group, then its commutator $\mathcal{D}G$ is a connected linear K -subgroup of G ([Spr98], Corollary 2.2.8).

Fix an algebraic closure \overline{K} of K .

Definition 2.4. A linear algebraic group G over K is

- (D1) finite if $G(\overline{K})$ is finite (this definition uses the fact that $\text{char}K = 0$).
- (D2) connected if its underlying topological space is connected, with respect to the Zariski topology.
- (D3) solvable if its derived series $G \supset \mathcal{D}G \supset \mathcal{D}^2G \supset \dots$ has \mathcal{D}^iG trivial for some $i \in \mathbb{Z}_{\geq 0}$.
- (D4) reductive if its geometric unipotent radical is trivial, i.e. if $R_u(G_{\overline{K}}^0) = R_u(G^0)_{\overline{K}}$ is trivial. Note that we don't require G to be connected.
- (D5) semisimple if its geometric radical is trivial, i.e. if $R(G_{\overline{K}}^0) = R(G^0)_{\overline{K}}$ is trivial. Again, note that we don't require G to be connected.
- (D6) of multiplicative type if $G_{\overline{K}}$ is \overline{K} -isomorphic to a closed subgroup of $\mathbb{G}_{m, \overline{K}}^N$, for some $N \in \mathbb{Z}_{\geq 0}$.
- (D7) a torus if $G_{\overline{K}}$ is \overline{K} -isomorphic to $\mathbb{G}_{m, \overline{K}}^N$, for some $N \in \mathbb{Z}_{\geq 0}$.
- (D8) unipotent if $G_{\overline{K}}$ admits a (finite) composition series over \overline{K} such that each successive quotient is isomorphic (over \overline{K}) to a closed subgroup of $\mathbb{G}_{a, \overline{K}}$; equivalently (cf. [DGA] XVII, Théorème 3.5), if G admits a (finite) central series over K such that each successive quotient is isomorphic (over K) to \mathbb{G}_a (here we use the fact that $\text{char}K = 0$); equivalently, if $G = R_u(G)$. In particular, if G is unipotent, then G is connected. Note that “being unipotent” is a geometric property, i.e. is stable under base-extending K (cf. [DGA] XVII, Proposition 2.2(i)).

We define the following sets:

$$\begin{aligned}
\mathcal{L}_K &= \{G : G \text{ is a linear algebraic } K\text{-group}\} / \sim, \\
\mathcal{F}_K &= \{G \in \mathcal{L}_K : G \text{ is finite}\}, \\
\mathcal{C}_K &= \{G \in \mathcal{L}_K : G \text{ is connected}\}, \\
\mathcal{Ab}_K &= \{G \in \mathcal{L}_K : G \text{ is commutative}\}, \\
\text{Sol}_K &= \{G \in \mathcal{L}_K : G \text{ is solvable}\}, \\
\mathcal{M}_K &= \{G \in \mathcal{L}_K : G \text{ is of multiplicative type}\}, \\
\mathcal{T}_K &= \{G \in \mathcal{L}_K : G \text{ is a torus}\}, \\
\mathcal{R}_K &= \{G \in \mathcal{L}_K : G \text{ is reductive}\}, \\
\mathcal{RR}_K &= \{G \in \mathcal{R}_K : H \leq_K G \Rightarrow H \in \mathcal{R}_K\}, \\
\mathcal{U}_K &= \{G \in \mathcal{L}_K : G \text{ is unipotent}\},
\end{aligned}$$

where $G_1 \sim G_2$ if and only if G_1 is k -isomorphic to G_2 as a k -group.

Notation. For any $\mathcal{S} \subset \mathcal{L}_K$, we write \mathcal{S}^{Sol} for $\mathcal{S} \cap \text{Sol}_K$, \mathcal{S}^{Ab} for $\mathcal{S} \cap \mathcal{Ab}_K$, and so on.

Recall that, if A, B , and G are linear algebraic groups over K , we say that G is an extension of A by B if G fits into a short exact sequence $1 \rightarrow B \rightarrow G \rightarrow A \rightarrow 1$ of linear algebraic K -groups.

Definition 2.5. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_K$. We define

$$\text{Ext}(\mathcal{A}, \mathcal{B}) = \{G \in \mathcal{L}_K : G \text{ is an extension of } A \text{ by } B, \text{ for some } A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$

Remark 2.6. If $G \in \text{Ext}(\mathcal{A}, \mathcal{B})$, then, by definition, G fits into a short exact sequence $1 \rightarrow B \rightarrow G \rightarrow A \rightarrow 1$, for some $B \in \mathcal{B}$ and $A \in \mathcal{A}$. If $(f, Y, G) \in \text{Ext}(\mathcal{A}, \mathcal{B})(X)$, then f naturally decomposes into $(Y, B) \in \mathcal{B}(Z)$ and $(Z, A) \in \mathcal{A}(X)$, where $Z := Y/B$ is the push-forward of $f : Y \rightarrow X$ along the morphism $G \rightarrow A$ (see §2.2 of [Sko01] for more details).

It might be worth mentioning that some of the sets defined above can be recast in terms of extensions of linear algebraic groups. For example, it is easy to check that $\mathcal{L}_K = \text{Ext}(\mathcal{F}_K, \mathcal{C}_K)$; perhaps less obviously, $\mathcal{RR}_K = \text{Ext}(\mathcal{F}_K, \mathcal{T}_K)$ (cf. Lemma 5.9).

2.2. Some properties of extensions. We describe here some properties of $\text{Ext}(\mathcal{A}, \mathcal{B})$ for the special cases when $\mathcal{A} = \mathcal{F}_K, \mathcal{F}_K^{\text{Sol}}$, or $\mathcal{F}_K^{\text{Ab}}$.

In general, we say that some $\mathcal{S} \subset \mathcal{L}_K$ is closed

- (E1) under taking direct products if $S_1, S_2 \in \mathcal{S}$ implies that $S_1 \times_K S_2 \in \mathcal{S}$;
- (E2) under taking (closed) K -subgroups if $S \in \mathcal{S}$ and $H \leq_K S$ imply that $H \in \mathcal{S}$;
- (E3) under taking K -twists if, whenever $S \in \mathcal{S}$ and $S' \in \mathcal{L}_K$ are such that $S'_K \cong S_{\overline{K}}$ as a \overline{K} -groups, then we have that $S' \in \mathcal{S}$;
- (E4) under “base changing/restricting” if $S \in \mathcal{S}$ and L/K a finite field extension imply that $\mathfrak{R}_{L/K}(S_L) \in \mathcal{S}$, where $\mathfrak{R}_{L/K}$ denotes the Weil restriction.

Remark 2.7. The sets $\mathcal{F}_K, \mathcal{F}_K^{\text{Sol}}$, and $\mathcal{F}_K^{\text{Ab}}$ satisfy all the properties above. Note, however, that $\mathcal{F}_K^{\text{Ab}}$ is not closed under extensions, while both \mathcal{F}_K and $\mathcal{F}_K^{\text{Sol}}$ are.

Lemma 2.8. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking closed K -subgroups, then so is $\text{Ext}(\mathcal{F}_K, \mathcal{B})$. The same is true with “ \mathcal{F}_K ” replaced by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”.*

Proof. Let $G \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$, and say it fits into the short exact sequence $1 \rightarrow B \xrightarrow{\beta} G \xrightarrow{\alpha} A \rightarrow 1$, for some $A \in \mathcal{F}_K$ and $B \in \mathcal{B}$. Let $H \leq_K G$. Consider the short exact sequence

$$1 \rightarrow \beta(B) \cap H \xrightarrow{\beta'} H \xrightarrow{\alpha'} \alpha(H) \rightarrow 1,$$

where α' and β' are the obvious maps. This is indeed exact at the ends, and to see exactness at the middle, just notice that $\beta(B) \cap H \cong \ker(\alpha) \cap H \cong \ker(\alpha')$. Since α is a K -homomorphism, it follows that $\alpha(H) \leq_K A$, and hence that $\alpha(H) \in \mathcal{F}_K$. Moreover, by assumption, \mathcal{B} is closed under taking K -subgroups, meaning that $\beta(B) \cap H \in \mathcal{B}$. Hence, $H \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$. The same proof works also if we replace “ \mathcal{F}_K ” by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”. \square

Lemma 2.9. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking direct products, then so is $\text{Ext}(\mathcal{F}_K, \mathcal{B})$. The same is true with “ \mathcal{F}_K ” replaced by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”.*

Proof. Easy. \square

Lemma 2.10. *Suppose $\text{Ext}(\mathcal{F}_K, \mathcal{B})$ is closed under taking direct products and closed K -subgroups. Let $G, G', G'' \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$, and suppose $\phi' : G' \rightarrow G$ and $\phi'' : G'' \rightarrow G$ are morphisms of algebraic K -groups defined. Then $G' \times_G G'' \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$, where the fibred product is with respect to ϕ and ϕ' . The same is true with “ \mathcal{F}_K ” replaced by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”.*

Proof. This follows from the assumption that $\text{Ext}(\mathcal{F}_K, \mathcal{B})$ is closed under taking (closed) K -subgroups and direct products, since $G' \times_G G'' \cong \ker(\phi : G' \times_K G'' \rightarrow G)$ with $\phi((g', g'')) = \phi'(g')\phi''^{-1}(g'')$ by Yoneda's lemma, and so $G' \times_G G''$ is a closed K -subgroup of $G' \times_K G''$. The same proof works also if we replace “ \mathcal{F}_K ” by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”. \square

Let L/K be a finite field extension. Recall that, for any quasi-projective L -scheme W , the Weil restriction $\mathfrak{R}_{L/K}(W)$ exists and represents the contravariant functor $T \mapsto W(T \times_K L)$ from \mathbf{Sch}_K to \mathbf{Set} . We list here some of the properties of $\mathfrak{R}_{L/K}$ (see [Vos11], Chap 1, §3.12).

(W1) If X is a K -variety, there is a functorial bijection

$$\text{Hom}_K(X, \mathfrak{R}_{L/K}(X \times_K L)) \cong \text{Hom}_L(X \times_K L, X \times_K L).$$

This gives, by considering $\text{id}_{X \times_K L} : X \times_K L \rightarrow X \times_K L$ on the right-hand side, the canonical embedding $\iota : X \rightarrow \mathfrak{R}_{L/K}(X \times_K L)$.

(W2) If X is affine (respectively, smooth) variety over L , then $\mathfrak{R}_{L/K}(X)$ is also affine (respectively, smooth). If G is an L -group, then $\mathfrak{R}_{L/K}(G)$ is a K -group. Moreover, $\mathfrak{R}_{L/K}(G)$ is connected if and only if G is connected, and if G is commutative, then so is $\mathfrak{R}_{L/K}(G)$.

(W3) Given a short exact sequence $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ of algebraic L -groups,

$$1 \rightarrow \mathfrak{R}_{L/K}(G') \rightarrow \mathfrak{R}_{L/K}(G) \rightarrow \mathfrak{R}_{L/K}(G'') \rightarrow 1$$

is a short exact sequence of algebraic K -groups. In other words, the functor $\mathfrak{R}_{L/K}(-)$ preserves short exactness.

Lemma 2.11. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under “base changing/restricting” and under taking (closed) K -subgroups, then $\text{Ext}(\mathcal{F}_K, \mathcal{B})$ is closed under K -twists. The same is true with “ \mathcal{F}_K ” replaced by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”.*

Proof. We need to show that if $G \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$ and $\tilde{G} \in \mathcal{L}_K$ is such that $\tilde{G}_{\bar{K}} \cong G_{\bar{K}}$ (over \bar{K}), then $\tilde{G} \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$. Consider the short exact sequence

$$1 \rightarrow B \xrightarrow{b} G \xrightarrow{a} A \rightarrow 1.$$

Let \tilde{G} be a K -twist of G , so that $\tilde{G}_{\bar{K}} \cong G_{\bar{K}}$ over \bar{K} . In particular, there is a finite field extension L/K such that $G_L \cong \tilde{G}_L$. Let $\phi : G_L \xrightarrow{\sim} \tilde{G}_L$ be an L -isomorphism of these L -groups, with $\varphi : \tilde{G}_L \xrightarrow{\sim} G_L$ as inverse. Since group schemes represent functors of fppf group sheaves, short exact sequences of group schemes are stable under base change. This gives a short exact sequence $1 \rightarrow B_L \xrightarrow{b_L} G_L \xrightarrow{a_L} A_L \rightarrow 1$, which, using the L -isomorphisms ϕ and φ above, in turn gives the short exact sequence (as L -groups)

$$(2.1) \quad 1 \rightarrow B_L \xrightarrow{\phi \circ b_L} \tilde{G}_L \xrightarrow{a_L \circ \varphi} A_L \rightarrow 1.$$

Applying $\mathfrak{R}_{L/K}(-)$ to (2.1) gives a short exact sequence of algebraic K -groups

$$1 \rightarrow \mathfrak{R}_{L/K}(B_L) \xrightarrow{\beta} \mathfrak{R}_{L/K}(\tilde{G}_L) \xrightarrow{\alpha} \mathfrak{R}_{L/K}(A_L) \rightarrow 1.$$

Now, by assumption $\mathfrak{R}_{L/K}(B_L) \in \mathcal{B}$, and clearly $\mathfrak{R}_{L/K}(A_L) \in \mathcal{F}_K$. Moreover, $\iota : \tilde{G} \rightarrow \mathfrak{R}_{L/K}(\tilde{G}_L)$ is an embedding. Hence, we can consider the short exact sequence

$$1 \rightarrow \mathfrak{R}_{L/K}(B_L) \cap \iota(\tilde{G}) \xrightarrow{\beta'} \iota(\tilde{G}) \cong \tilde{G} \xrightarrow{\alpha'} \alpha(\iota(\tilde{G})) \rightarrow 1,$$

where α' and β' are the obvious maps. Since \mathcal{B} is closed under taking (closed) K -subgroups, and since $\alpha(\iota(\tilde{G})) \leq_K \mathfrak{R}_{L/K}(A_L)$, meaning that $\alpha(\iota(\tilde{G})) \in \mathcal{F}_K$, we have that $\tilde{G} \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$, as required. The same proof works also if we replace “ \mathcal{F}_K ” by “ $\mathcal{F}_K^{\text{Sol}}$ ” or by “ $\mathcal{F}_K^{\text{Ab}}$ ”. \square

Let G be an abstract group. By the constant K -group \tilde{G} induced by G we mean the disjoint union of copies of $\text{Spec}K$, indexed by G , together with the obvious group structure induced by G . If W is a K -scheme, by a (left) action of the abstract group G on W we mean a group homomorphism $\phi : G \rightarrow \text{Aut}_K(W)$. One can show that the group homomorphisms $\phi : G \rightarrow \text{Aut}_K(W)$ defining an action of G on W correspond precisely to the K -morphisms $\tilde{\phi} : \tilde{G} \times W \rightarrow W$ defining an action of \tilde{G} on W ; we will use these two notions of “action” interchangeably. For this reason, with abuse of notation, we will denote by “ G ” both the abstract group G and the constant K -group induced by G .

Lemma 2.12. *Suppose that $\mathcal{B} \subset \mathcal{L}_K$ is closed under direct products and that $1 \in \mathcal{B}$. Let $G \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$. Let F be a finite abstract group and, by abusing notation, let $F \in \mathcal{F}_K$ be the induced constant finite K -group. Let F act on $G^{|F|}$ (the direct product of G with itself $|F|$ times) by permuting the coordinates, and let $\phi : F \rightarrow \text{Aut}_K(G^{|F|})$ be the corresponding group homomorphism coming from this action. Then $G^{|F|} \rtimes_{\phi} F \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$. The same is true with “ \mathcal{F}_K ” replaced by “ $\mathcal{F}_K^{\text{Sol}}$ ”.*

Proof. Let $\phi : F \rightarrow \text{Aut}_K(G^{|F|})$ denote the morphism induced by the action of F (with F seen as a subgroup of the symmetric group $S_{|F|}$, by Cayley’s theorem). Since $G \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$, we have that G fits, say, into the short exact sequence

$$1 \rightarrow B \xrightarrow{b} G \xrightarrow{a} A \rightarrow 1,$$

where $B \in \mathcal{B}$ and $A \in \mathcal{F}_K$. By Lemma 2.9, we know that $G^{|F|} \in \text{Ext}(\mathcal{F}_K, \mathcal{B})$ and that it fits into the exact sequence

$$1 \rightarrow B^{|F|} \xrightarrow{b^{|F|} := (b, \dots, b)} G^{|F|} \xrightarrow{a^{|F|} := (a, \dots, a)} A^{|F|} \rightarrow 1,$$

with $A^{|F|} \in \mathcal{F}_K$ and $B^{|F|} \in \mathcal{B}$. Since $a^{|F|} : G^{|F|} \rightarrow A^{|F|}$ is surjective and its kernel $\ker(a^{|F|}) = (\ker(a), \dots, \ker(a)) \cong B^{|F|}$ is ϕ -characteristic (i.e. $\phi(\ker(a^{|F|})) = \ker(a^{|F|})$) as ϕ just acts by permuting the coordinates, there is a $\varphi : F \rightarrow \text{Aut}_K(A)$ making the following diagram commute

$$\begin{array}{ccc} G^{|F|} & \xrightarrow{\phi} & G^{|F|} \\ a^{|F|} \downarrow & & \downarrow a^{|F|} \\ A^{|F|} & \xrightarrow{\varphi} & A^{|F|} \end{array}$$

In particular, this means that, for any $f \in F$ and $g \in G^{|F|}$, we have

$$(2.2) \quad a^{|F|}(\phi_f(g)) = \varphi_f(a^{|F|}(g)),$$

where $\phi_f := \phi(f)$ and $\varphi_f := \varphi(f)$.

Since, by assumption, $1 \in \mathcal{B}$, we have that F is in $\text{Ext}(\mathcal{F}_K, \mathcal{B})$, as it fits into the short exact sequence $1 \rightarrow 1 \rightarrow F \xrightarrow{\text{id}} F \rightarrow 1$. Now consider the sequence

$$1 \rightarrow \underbrace{B^{|F|} \rtimes_{\phi} 1}_{\cong B^{|F|}} \xrightarrow{(b^{|F|}, 1)} G^{|F|} \rtimes_{\phi} F \xrightarrow{(a^{|F|}, \text{id})} A^{|F|} \rtimes_{\varphi} F \rightarrow 1,$$

where $B^{|F|} \rtimes_{\phi} 1 \cong B^{|F|} \in \mathcal{B}$ and $A^{|F|} \rtimes_{\varphi} F \in \mathcal{F}_K$, since \mathcal{F}_K is closed under extensions. Clearly, $(b^{|F|}, 1)$ is a K -group homomorphism. One can check that $(a^{|F|}, \text{id})$ is also a K -group homomorphism, given our choice of φ and (2.2). Finally, it is clear that the sequence is exact, since the maps come from two short exact sequences. The same proof works also if we replace “ \mathcal{F}_K ” by “ $\mathcal{F}_K^{\text{Sol}}$ ”. \square

3. STATEMENT OF THE FIRST RESULT

Let k be a number field and let X be a nice over k . We define

$$(3.1) \quad X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} := \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z] \in H^1(X, F)} \bigcup_{[\tau] \in H^1(k, F)} f^{\tau}(Z^{\tau}(\mathbb{A}_k)^{\mathcal{O}}),$$

where $\mathcal{O} : \mathbf{Sch}_k^{\text{opp}} \rightarrow \mathbf{Sets}$ be any functor such that $X(\mathbb{A}_k)^{\mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Br}_1}$; the “solvable” and “abelian” versions of $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ are defined by replacing “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ” and “ $\mathcal{F}_k^{\text{Ab}}$ ”, respectively. We suppose further that there exists a set $\mathcal{S}_k \subset \mathcal{L}_k$ such that the following three conditions hold.

Condition 1. \mathcal{S}_k is contained in \mathcal{C}_k , closed under k -twists, and such that, for any nice k -variety W , we have $W(\mathbb{A}_k)^{\mathcal{O}} \subset W(\mathbb{A}_k)^{\mathcal{S}_k}$.

Condition 2. For any nice k -variety W' , we have $W'(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \subset W'(\mathbb{A}_k)^{\mathcal{O}}$. The “solvable” and “abelian” versions of this condition are obtained by replacing “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ” and “ $\mathcal{F}_k^{\text{Ab}}$ ”, respectively.

Condition 3. $\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking closed k -subgroups, under taking k -twists, under taking direct products, and such that conclusion of Lemma 2.12 (with $\mathcal{B} = \mathcal{S}_k$) holds. The “solvable” and “abelian” versions of this condition are obtained by replacing “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ” and “ $\mathcal{F}_k^{\text{Ab}}$ ”, respectively.

The following theorem is then a generalisation of the results in [Dem09] and [Sko09].

Theorem 3.1. *Let X be a nice variety over k . Let $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ be defined as above, and assume Conditions 1, 2, and 3 hold. Then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}.$$

If we consider the “solvable version” of $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, and the “solvable versions” of Conditions 1, 2, and 3 hold, then the result is also true if we replace “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ”.

Remark 3.2. Notice that $\mathcal{F}_k \subset \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ and $\mathcal{S}_k \subset \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, and the same holds if we replace “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ”.

4. PROOF OF THEOREM 3.1

In this section, we give a proof of Theorem 3.1. We will prove the “standard version” of the theorem, but for the “solvable version”, the proof is identical, modulo replacing “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ” when necessary. Most of the results in this section are just restatements of results in [Dem09] and [Sko09]; we often try to sketch the proofs of these results, for the reader’s convenience. We remark, however, that we use a *finesse* – Proposition 4.14 – not present (and not needed) in [Dem09] and [Sko09], which gives us flexibility in the applications of Theorem 3.1 (see §5).

4.1. Proof that $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \supset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$. In this subsection, it suffices to assume that \mathcal{S}_k satisfies Conditions 1 and 3 (without Lemma 2.12).

Lemma 4.1. *Let X be a smooth, projective variety over a number field k . Let $(f, Z, F) \in \mathcal{F}_k(X)$. Then Z is smooth and projective.*

Proof. This follows as f is finite and étale. \square

Definition 4.2. Let $(Y, G), (Z, H) \in \mathcal{L}_k(X)$. An X -torsor morphism $f = (\phi, \psi) : (Y, G) \rightarrow (Z, H)$ is a pair (ϕ, ψ) where $\phi : Y \rightarrow Z$ is an X -morphism and $\psi : G \rightarrow H$ is a homomorphism, compatible in the sense that the diagram

$$\begin{array}{ccc} Y \times_X G_X & \xrightarrow{\phi \times \psi} & Z \times_X H_X \\ \downarrow & \searrow \phi & \downarrow \\ Y & & Z \\ & \searrow & \swarrow \\ & & X \end{array}$$

commutes. We say that the X -torsor morphism $f = (\phi, \psi)$ is **surjective** if ϕ (equivalently, ψ) is surjective.

Remark 4.3. If $f = (\phi, \psi) : (Y, G) \rightarrow (Z, H)$ is surjective, then $Y \rightarrow Z$ is a $(\ker \psi)$ -torsor over Z .

Proposition 4.4 ([Dem09] Lemme 3). *Let X be a nice variety over a number field k . Let $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ and let $(f, Z, F) \in \mathcal{F}_k(X)$. Then there exist:*

- (i) an $F' \in \mathcal{F}_k$,
- (ii) a $(X', F') \in \mathcal{F}_k(X)$ with X' geometrically integral,
- (iii) a 1-cocycle $\sigma \in Z^1(k, F)$,
- (iv) a morphism of X -torsors $(\phi, \psi) : (X', F') \rightarrow (Z^\sigma, F^\sigma)$

such that (P_v) lifts to some $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$.

Proof. The proof is similar as the one in [Dem09], being careful to replace “Br” and “ét,Br” by “ \mathcal{O} ” and “ $\mathcal{F}_k, \mathcal{O}$ ”, respectively. \square

The next proposition, which allows us to lift cocycles, is essentially due to Demarche ([Dem09], Proposition 4), with the caveat that, while he requires $(P_v) \in X(\mathbb{A}_k)^{\text{ét}, \text{Br}}$, we take $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$.

Proposition 4.5 (Demarche, [Dem09] Proposition 4). *Let X be a nice variety over a number field k . Let $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ and let $(f, Y, G) \in \mathcal{L}_k(X)$. Let*

$$1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$$

be a short exact sequence in \mathcal{L}_k , with $H \in \mathcal{C}_k$ and $F \in \mathcal{F}_k$. We denote the pushforward of $f : Y \rightarrow X$ under the morphism $G \rightarrow F$ by $(Z, F) \in \mathcal{F}_k(X)$. Let $\sigma \in Z^1(k, F)$ be a cocycle coming from Proposition 4.4 applied to the torsor $Z \rightarrow X$ and the point (P_v) . Then the class $[\sigma] \in H^1(k, F)$ lifts to a class $[\tau] \in H^1(k, G)$.

Remark 4.6. Let $H \in \mathcal{C}_k$. Since H contains a k -point (the identity), it is also geometrically connected (see [Gro65], 4.5.14).

Proof. In the proof of Proposition 4 in [Dem09], the only place where Demarche uses the fact that $(P_v) \in X(\mathbb{A}_k)^{\text{ét, Br}}$ is in a passage of the proof of “Le lemme 7 implique la proposition 4”. In this passage, Demarche wants to prove the existence of some torsor of type λ' . To do so, he appeals to Corollary 6.1.3 (1) in [Sko01]. Indeed, he knows that (P_v) lifts to some $(Q_v) \in X'(\mathbb{A}_k)^{\text{Br}}$ for some nice X' ; this means in particular that $X'(\mathbb{A}_k)^{\text{Br}_1} \neq \emptyset$. But X' is proper and geometrically integral (and, in particular, $\bar{k}[X']^* = \bar{k}^*$), and $\emptyset \neq X'(\mathbb{A}_k)^{\text{Br}_1} = \bigcap_{\lambda} X'(\mathbb{A}_k)^{\text{Br}_{\lambda}}$ (see [Sko01] for the definitions). Hence, since $X'(\mathbb{A}_k)^{\text{Br}_{\lambda'}} \neq \emptyset$, the required torsor of type λ' exists by Corollary 6.1.3 (1) in [Sko01].

In our case, $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$. By Proposition 4.4, (P_v) lifts to some $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$ for some nice X' . Hence, $\emptyset \neq X'(\mathbb{A}_k)^{\mathcal{O}} \subset X'(\mathbb{A}_k)^{\text{Br}_1}$ (where the inclusion holds by assumption on \mathcal{O}), and since X' is proper and geometrically integral, we can appeal to Corollary 6.1.3 (1) in [Sko01] as well to conclude that the relevant torsor of type λ' exists. All the other parts of the proof of Proposition 4 in [Dem09] are unchanged. \square

We have an analogue of Théorème 1 in [Dem09].

Theorem 4.7. *Let X be a nice variety over k . Then $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$.*

Proof. Let $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$. We fix $G \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ and $(g, Y, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$. Then G fits, say, into the short exact sequence $1 \rightarrow S \rightarrow G \rightarrow F \rightarrow 1$, for some $F \in \mathcal{F}_k$ and $S \in \mathcal{S}_k$. By pushing forward g along $G \rightarrow F$, we can decompose $Y \rightarrow X$ into $(Y, S) \in \mathcal{S}_k(X)$ and $(Z, F) \in \mathcal{F}_k(X)$. By applying Proposition 4.4 to (Z, F) and (P_v) , we get an $F' \in \mathcal{F}_k$, an $(X', F') \in \mathcal{F}_k(X)$ with X' geometrically integral, a 1-cocycle $\rho \in Z^1(k, F)$, and an X -torsor morphism $\phi : X' \rightarrow Z^{\rho}$ such that (P_v) lifts to a point $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$. Hence, we have the commutative triangle

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & Z^{\rho} \\ & \searrow \begin{array}{l} F' \\ f' \end{array} & \downarrow F^{\rho} \\ & & X \end{array}$$

We now apply Proposition 4.5 to the X -torsor (g, Y, G) , the short exact sequence $1 \rightarrow S \rightarrow G \rightarrow F \rightarrow 1$ (note here that $S \in \mathcal{C}_k$, as $\mathcal{S}_k \subset \mathcal{C}_k$ by Condition 1), and $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ to conclude that $[\rho] \in H^1(k, F)$ lifts to some $[\mu] \in H^1(k, G)$. It follows that the G^{μ} -torsor $g^{\mu} : Y^{\mu} \rightarrow X$ can be decomposed naturally into the S^{μ} -torsor $Y^{\mu} \rightarrow Z^{\rho}$ and the F^{ρ} -torsor $Z^{\rho} \rightarrow X$. Now consider the fibred product $Y^{\mu} \times_{Z^{\rho}} X'$. This is naturally a S^{μ} -torsor over X' , and we remark that $(p_2, Y^{\mu} \times_{Z^{\rho}} X', S^{\mu}) \in \mathcal{S}_k(X')$ as \mathcal{S}_k is closed under k -twists. The following diagram summarises the constructions done so far:

$$\begin{array}{ccc} Y^{\mu} \times_{Z^{\rho}} X' & \xrightarrow{p_1} & Y^{\mu} \\ p_2 \downarrow S^{\mu} & & \downarrow S^{\mu} \\ X' & \xrightarrow{\phi} & Z^{\rho} \\ & \searrow \begin{array}{l} F' \\ f' \end{array} & \downarrow F^{\rho} \\ & & X \end{array} \quad \begin{array}{l} \curvearrowright \\ g^{\mu} \\ \curvearrowleft \end{array} G^{\mu}$$

Note that, since X is smooth and projective and $X' \rightarrow X$ is finite and étale, then X' is also smooth and projective; moreover, since X' is geometrically connected, it follows that X' is nice. Hence, we can use Condition 1 to deduce that

$$X'(\mathbb{A}_k)^{\mathcal{O}} \subset \bigcup_{[\nu] \in H^1(k, S^\mu)} p_2^\nu((Y^\mu \times_{Z^\rho} X')^\nu(\mathbb{A}_k)).$$

In particular, there is some $\nu \in Z^1(k, S^\mu)$ such that (Q_ν) lifts to $(R_\nu) \in (Y^\mu \times_{Z^\rho} X')^\nu(\mathbb{A}_k)$. Then, arguing in an identical manner as in Théorème 1 in [Dem09], we can conclude that

$$(P_\nu) \in \bigcup_{[\tau] \in H^1(k, G)} g^\tau(Y^\tau(\mathbb{A}_k)).$$

Since $(P_\nu) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, $G \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, and (g, Y, G) were arbitrary, we get that

$$X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)},$$

as required. \square

Remark 4.8. In this section, we only use Condition 1 and Condition 3 without Lemma 2.12, and these hold if we replace “ \mathcal{F}_k ” with “ $\mathcal{F}_k^{\text{Ab}}$ ”. Hence, we can also conclude that $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Ab}}, \mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Ab}}, \mathcal{S}_k)}$.

4.2. Proof that $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \subset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$. In this subsection, we assume that \mathcal{S}_k satisfies Conditions 1, 2, 3.

Proposition 4.9 (Based on [Sto07], Proposition 5.17). *Let X be a smooth, proper variety over a number field k . Let $(Y, F) \in \mathcal{F}_k(X)$. For any $(P_\nu) \in X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$, there exists a twist $(Y', F') \in \mathcal{F}_k(X)$ of (Y, F) with the following property. For any surjective X -torsor morphism $(\phi, \psi) : (Z, G) \rightarrow (Y', F')$, where $(Z, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$, there exists a twist $Z' \rightarrow Y'$ of $(Z, \ker \psi) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(Y')$ such that (P_ν) lifts to a point in $Z'(\mathbb{A}_k)$.*

Proof of Proposition 4.9. We follow closely the first part of the proof of Proposition 5.17 in [Sto07], modifying it accordingly when needed.

Let $P \in X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$. Let $(Y, F) \in \mathcal{F}_k(X)$. Since X is proper, by ([Sko01], Proposition 5.3.2) there are only finitely many twists (Y^σ, F^σ) of (Y, F) such that $Y^\sigma(\mathbb{A}_k) \neq \emptyset$. Moreover, since $P \in X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$ and $\mathcal{F}_k \subset \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, we have that P lifts to some point in some $Y^\sigma(\mathbb{A})$. Let Y_1, \dots, Y_s be the finitely many twists of (Y, F) such that P lifts to some adelic point in them.

Let $\tau(j) \subset \{1, \dots, s\}$ be the set of indices i such that, for every $(Z, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ mapping to $Y_j \rightarrow X$, there is a twist Z^ξ that lifts P and induces a twist of Y_j that is isomorphic to Y_i . Following Stoll, one can show that

- (i) $\tau(j) \neq \emptyset$ (using Condition 3);
- (ii) if $i \in \tau(j)$, then $\tau(i) \subset \tau(j)$;
- (iii) for some j , we have $j \in \tau(j)$.

We can take Y' to be Y_j , where $j \in \tau(j)$. \square

Remark 4.10. For the above proof to make sense, we need $\mathcal{F}_k \subset \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$.

Lemma 4.11 (Skorobogatov, [Sko09] Corollary 2.7). *Let Y be a proper variety over a number field k , and let $(g, Z, G) \in \mathcal{L}_k(Y)$. Then the set $Y(\mathbb{A}_k)^g$ is closed in $Y(\mathbb{A}_k)$. \square*

The following proposition is essentially due to Skorobogatov.

Proposition 4.12 (Skorobogatov, [Sko09] Proposition 2.3). *Let X be a variety over a number field k . Let $(Y, F) \in \mathcal{F}_k(X)$ and $(Z, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(Y)$. Then there exist a $(V, G') \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ and a surjective X -torsor morphism $(\theta, \psi) : (V, G') \rightarrow (Y, F)$ such that there is a surjective Y -torsor morphism $(\Theta, \Psi) : (V, \ker \psi) \rightarrow (Z, G)$ with $\ker \Psi \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$.*

Proof. As in the proof of Proposition 2.3 of [Sko09], let $V := \mathfrak{R}_{Y/X}(Z) \times_X Y$, where $\mathfrak{R}_{Y/X}(Z)$ is the Weil restriction (which exists since $Y \rightarrow X$ is finite étale, cf. §7.6 of [BLR90]), and let $\theta : V \rightarrow Y$ be the second projection. Consider the group $G^m \rtimes_{\rho} F(\bar{k})$, where $m = |F(\bar{k})|$ and where ρ is the action of the constant group $F(\bar{k})$ on G^m by permutation of coordinates. Since $G \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, by Condition 3 we have also that $G^m \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. Moreover, since $F(\bar{k}) \in \mathcal{F}_k$ is a constant group acting on G^m by permuting the coordinates, and since we are assuming Condition 3, by Lemma 2.12 we know that $G^m \rtimes_{\rho} F(\bar{k}) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. Now consider the k -twist of $G^m \rtimes_{\rho} F(\bar{k})$ given by $G' := \mathfrak{R}_{F/\text{Spec}k}(G_F) \rtimes_{\phi} F$, where ϕ is the action induced by ρ (and where $\mathfrak{R}_{F/\text{Spec}k}(G_F)$ exists, again cf. §7.6 of [BLR90]). Since, by Condition 3, $\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking k -twists, it follows that $G' \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. Following the proof of Proposition 2.3 of [Sko09], one can show that $V \rightarrow X$ is a G' -torsor, that θ and the natural projection $\psi : G' \rightarrow F$ together give a surjective X -torsor morphism $(\theta, \psi) : (V, G') \rightarrow (Y, F)$, and that there exists a surjective Y -torsor morphism $(\Theta, \Psi) : (V, \ker \psi) \rightarrow (Z, G)$. Moreover, since $\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking (closed) k -subgroups (by Condition 3), we also have that $\ker \psi \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. \square

We can now prove that $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$ is “well-behaved” with respect to torsors under groups in \mathcal{F}_k . Our proof is almost verbatim that of Theorem 1.1 in [Sko09].

Theorem 4.13. *For any $F \in \mathcal{F}_k$ and any $[Z] \in H^1(X, F)$, we have*

$$X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \bigcup_{[\tau] \in H^1(k, F)} f^{\tau}(Z^{\tau}(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}).$$

Proof. The inclusion “ \supset ” follows by pulling back torsors. Indeed, suppose that

$$(P_v) \in \bigcup_{[\tau] \in H^1(k, F)} f^{\tau}(Z^{\tau}(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}),$$

say $(P_v) = f^{\tau}((Q_v))$ for some $(Q_v) \in Z^{\tau}(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$ and some $[\tau] \in H^1(k, F)$. Then, if $(g, Y, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$, we obtain the torsor $(p_1, Z^{\tau} \times_X Y, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(Z^{\tau})$ by pulling back $(g, Y, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ along $f^{\tau} : Z^{\tau} \rightarrow X$. Since $(Q_v) \in Z^{\tau}(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$ and $G \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, there is some $[\mu] \in H^1(k, G)$ such that (Q_v) lifts to some $(R_v) \in (Z^{\tau} \times_X Y)^{\mu}(\mathbb{A}_k) = (Z^{\tau} \times_X Y^{\mu})(\mathbb{A}_k)$. By the commutativity of obvious pullback diagram, we conclude that $(P_v) \in g^{\mu}(Y^{\mu}(\mathbb{A}_k))$. Since $(g, Y, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ was arbitrary, it follows that $(P_v) \in X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$.

For the other inclusion, let $(P_v) \in X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$. Let $(\tilde{f}, \tilde{Z}, \tilde{F}) \in \mathcal{F}_k(X)$ be the twist of $(f, Z, F) \in \mathcal{F}_k(X)$ coming from Proposition 4.9. It is clear that we only need to show that (P_v) lifts to a point in $\tilde{Z}(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$. Suppose not, for a contradiction. Then

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{(g', W', G') \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(\tilde{Z})} \tilde{Z}(\mathbb{A}_k) \setminus \tilde{Z}(\mathbb{A}_k)^{g'},$$

where the cover on the right hand side is an open cover since each $\tilde{Z}(\mathbb{A}_k)^{g'}$ is closed in $\tilde{Z}(\mathbb{A}_k)$, by Lemma 4.11. Using compactness, we have

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{i=1}^n \tilde{Z}(\mathbb{A}_k) \setminus \tilde{Z}(\mathbb{A}_k)^{g_i},$$

for some $(g_i, W_i, G_i) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(\tilde{Z})$. Let $(g, W, G) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(\tilde{Z})$ denote the fibred product of the g_i over \tilde{Z} ; here $G \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, since $\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under finite fibred products. It is clear from the construction that $\tilde{f}^{-1}((P_v)) \cap \tilde{Z}(\mathbb{A}_k)^g = \emptyset$. By Proposition 4.12, there is a torsor $(V, L) \in \text{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ and a surjective X -torsor morphism $(\theta, \psi) : (V, L) \rightarrow (\tilde{Z}, \tilde{F})$ such that there is a surjective \tilde{Z} -torsor morphism $(V, \ker \psi) \rightarrow (W, G)$. To get the required contradiction, one can then argue identically as in the proof of Theorem 1.1 in [Sko09]. \square

Proposition 4.14. *Let X be a nice variety over a number field k . If $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$, then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} = \bigcap_{F \in \mathcal{F}_k} \bigcap_{\substack{[Z] \in H^1(X, F) \\ Z \text{ nice}}} \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O}),$$

that is, we can restrict our attention to nice X -torsors $[Z] \in H^1(X, F)$ only. Similarly if we replace “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ”.

Proof. The inclusion “ \subset ” is clear, so we just prove the opposite inclusion.

Let $(P_v) \in \bigcap_F \bigcap_{[Z] \text{ nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O})$. Fix $F \in \mathcal{F}_k$ and $[f : Z \rightarrow X] \in H^1(X, F)$. We need to show that $(P_v) \in \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O})$. Consider $(\bar{Z}, \bar{F}) \in \mathcal{F}_k(\bar{X})$. Let \bar{Z}_0 be a connected component of \bar{Z} , and let $\bar{F}_0 \leq_{\bar{k}} \bar{F}$ be its stabiliser. Then we have an \bar{X} -torsor morphism $(\bar{Z}_0, \bar{F}_0) \rightarrow (\bar{Z}, \bar{F})$, given by inclusion.

By the notion of cofinal coverings in [Sto07] and by Lemma 5.7 in [Sto07], using our assumption that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$, we know that there exist a $F' \in \mathcal{F}_k$ and a $(f', W, F') \in \mathcal{F}_k(X)$ with W geometrically connected such that there is an \bar{X} -torsor morphism $(\bar{W}, \bar{F}') \rightarrow (\bar{Z}_0, \bar{F}_0)$. Hence, by composition, we get an \bar{X} -torsor morphism

$$(\bar{W}, \bar{F}') \rightarrow (\bar{Z}_0, \bar{F}_0) \rightarrow (\bar{Z}, \bar{F}),$$

with W geometrically connected. We can then apply Lemma 5.6 of [Sto07] to conclude that there exists a twist $(f^\tau, Z^\tau, F^\tau) \in \mathcal{F}_k(X)$ of (f, Z, F) such that there is an X -torsor morphism

$$(W, F') \rightarrow (Z^\tau, F^\tau),$$

with W geometrically connected. Since W is also smooth and projective (cf. Lemma 4.1), then W is nice, and the same is true for all its twists.

Without loss of generality (twisting if necessary), we can assume that (P_v) lifts to some point in $W(\mathbb{A}_k)^\mathcal{O}$. We can then use Lemma 5.6 in [Sto07] and the functoriality of \mathcal{O} to deduce that $(P_v) \in \bigcup_{\tau \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O})$. Since F and f were arbitrary, we conclude that

$$\bigcap_F \bigcap_{[Z] \text{ nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O}) \subset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}.$$

Hence, $\bigcap_F \bigcap_{[Z] \text{ nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^\mathcal{O}) = X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, meaning that we can restrict our attention to nice torsors only. Similarly for the “solvable version”. \square

Proof of Theorem 3.1. By Theorem 4.7, $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$.

If $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \emptyset$, then trivially $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \subset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, and we are done. So suppose $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \neq \emptyset$. In particular, this implies that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$. By Theorem 4.13, for any $[Z] \in H^1(X, F)$ and any $F \in \mathcal{F}_k$, we have

$$X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}).$$

Hence,

$$\begin{aligned} X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)} &= \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z] \text{ nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}) \\ &\subset \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z] \text{ nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}) \\ &= X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}, \end{aligned}$$

where the inclusion in the second line comes from Condition 2, and the equality in the bottom line comes from Proposition 4.14 (and the fact that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$).

A similar proof holds if we replace “ \mathcal{F}_k ” by “ $\mathcal{F}_k^{\text{Sol}}$ ”. \square

Remark 4.15. Proposition 4.14 makes Theorem 3.1 robust enough for applications. Indeed, we usually only have Condition 2 as we have stated it, and not a more general version as, for example, is the case for the proof of $X(\mathbb{A}_k)^{\text{Desc}} \subset X(\mathbb{A}_k)^{\text{ét, Br}}$ in [Sko09]: there, as a corollary to a result by Gabber, one has that $W(\mathbb{A}_k)^{\mathcal{C}_k} \subset W(\mathbb{A}_k)^{\text{ét, Br}}$ for *all* smooth, projective varieties over k (cf. Lemma 2.8 in [Sko09]). Proposition 4.14 tells us that, in fact, our version of Condition 2 is sufficient.

5. SOME APPLICATIONS OF THEOREM 3.1

5.1. The étale-Brauer set and its variations. Let X be a nice variety over k . Consider the “solvable” étale-Brauer set

$$X(\mathbb{A}_k)^{\text{Sol, Br}} := \bigcap_{F \in \mathcal{F}_k^{\text{Sol}}} \bigcap_{[Z] \in H^1(X, F)} \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\text{Br}}).$$

In our terminology, $\mathcal{O} = \text{Br}$. As we have seen in the introduction, for any nice variety W over k we have that $W(\mathbb{A}_k)^{\text{Br}} = W(\mathbb{A}_k)^{\mathcal{C}_k}$. Hence, in our setting, we can take $\mathcal{S}_k = \mathcal{C}_k$. As Condition 1 is clear, we only need to check that the “solvable version” of Conditions 2 and 3 hold. The latter condition is easy to see by the closure properties of $\mathcal{F}_k^{\text{Sol}}$, and the former one holds since $\mathcal{C}_k \subset \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)$. Hence, by applying the “solvable version” of Theorem 3.1, we can deduce the following.

Theorem 5.1. *Let X be a nice variety over k . Then $X(\mathbb{A}_k)^{\text{Sol, Br}} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)}$. \square*

5.2. The algebraic étale-Brauer set and its variations. Let X be a nice variety over k . Consider the algebraic étale-Brauer set

$$X(\mathbb{A}_k)^{\text{ét, Br}_1} := \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z] \in H^1(X, F)} \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\text{Br}_1}).$$

In our terminology, $\mathcal{O} = \text{Br}_1$.

Theorem 5.2 ([Sko01] Theorem 6.1.1). *Let Z be a variety over a number field k such that $\bar{k}[Z]^* = \bar{k}^*$. Then*

$$(5.1) \quad Z(\mathbb{A}_k)^{\text{Br}_1} = \bigcap_{\lambda: \hat{S} \rightarrow \text{Pic}(\bar{Z}) \text{ type}(Y, f) = \lambda} \bigcup f(Y(\mathbb{A}_k)),$$

where \hat{S} ranges over the finitely generated $\text{Gal}(\bar{k}/k)$ -submodules of $\text{Pic}(\bar{X})$.

Remark 5.3. We recall that the type λ of a torsor $(f, Y, S) \in \mathcal{M}_k(Z)$ is defined by $\chi([Y]) =: \lambda$, where χ is the map in the fundamental exact sequence due to Colliot-Thélène and Sansuc (see [Sko01])

$$0 \rightarrow H^1(k, S) \rightarrow H^1(Z, S) \xrightarrow{\chi} \text{Hom}_{\text{Gal}(\bar{k}/k)}(\hat{S}, \text{Pic}(\bar{Z})) \xrightarrow{\partial} H^2(k, S) \rightarrow H^2(Z, S),$$

where $\hat{S} := \text{Hom}_{\bar{k}\text{-groups}}(\bar{S}, \mathbb{G}_{m, \bar{k}})$ is the module of characters of S . We remark that there is an anti-equivalence of categories

$$\left\{ \begin{array}{l} \text{algebraic linear } k\text{-groups} \\ \text{of multiplicative type} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{finitely generated } \mathbb{Z}\text{-modules with} \\ \text{a continuous action of } \text{Gal}(\bar{k}/k) \end{array} \right\}$$

given by the functor $S \mapsto \hat{S}$, with inverse given by $M \mapsto \text{Spec}(\bar{k}[M]^{\text{Gal}(\bar{k}/k)})$.

In particular, Theorem 5.2 tells us that, for any nice variety W over k , we have $W(\mathbb{A}_k)^{\mathcal{M}_k} = W(\mathbb{A}_k)^{\text{Br}_1}$. A sensible idea would be to take $\mathcal{S}_k = \mathcal{M}_k$, but unfortunately $\mathcal{M}_k \not\subset \mathcal{C}_k$, so Condition 1 fails. We take instead $\mathcal{S}_k = \mathcal{T}_k \subset \mathcal{C}_k$, and consider $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. Since our aim is to apply Theorem 3.1, we check that Conditions 1, 2, and 3 hold for our choice of \mathcal{S}_k .

Condition 1. This holds since $\mathcal{T}_k \subset \mathcal{C}_k$, the k -twist of an algebraic torus is again an algebraic torus ("being an algebraic torus" is a geometric condition), and since Proposition 5.2 (which tells us that $W(\mathbb{A}_k)^{\text{Br}_1} \subset W(\mathbb{A}_k)^{\mathcal{T}_k}$) holds.

Condition 2. As we have seen, $W(\mathbb{A}_k)^{\mathcal{M}_k} \subset W(\mathbb{A}_k)^{\text{Br}_1}$. But, by definition of groups of multiplicative type,

$$\mathcal{M}_k = \text{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathcal{A}b_k \subset \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) \subset \text{Ext}(\mathcal{F}_k, \mathcal{T}_k),$$

where, in the first inclusion, we have used the fact that $\mathcal{T}_k \subset \mathcal{A}b_k \subset \text{Sol}_k$ and that Sol_k is closed under taking quotients. Hence, Condition 2 and its "solvable version" also hold.

Condition 3. Notice that \mathcal{T}_k is clearly closed under taking finite direct products and that it contains the trivial group. By Lemma 2.9, we have that $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking direct products; moreover, the hypotheses of Lemma 2.12 hold (with $\mathcal{B} = \mathcal{T}_k$), meaning that the conclusion of Lemma 2.12 also holds. The same is true for $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$. Hence, it remains to check that $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$ are closed under taking closed k -subgroups and k -twists.

Lemma 5.4. $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ and $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{M}_k)$.

Proof. We first prove that $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$. The inclusion " \subset " is clear. For the other inclusion, let $G \in \text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ fit into the short exact sequence $1 \rightarrow M \rightarrow G \rightarrow F \rightarrow 1$, with $F \in \mathcal{F}_k$ and $M \in \mathcal{M}_k$. But M contains a maximal subtorus T (cf. [DGA], SGA3.IV, §1.3), which is a normal subgroup of G (since T is characteristic in M) and with $G/T \in \mathcal{F}_k$. Hence, G fits into the short exact sequence

$$1 \rightarrow T \rightarrow G \rightarrow G/T \rightarrow 1,$$

with $T \in \mathcal{T}_k$ and $G/T \in \mathcal{F}_k$.

We now prove that $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{M}_k)$. Using the closure properties of solvable groups, we have $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \text{Sol}_k$ and $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{M}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{M}_k) \cap \text{Sol}_k$. But we have just seen that $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$; by intersecting both sides with Sol_k , the result follows. \square

Lemma 5.5. *$\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under k -subgroups. The same holds for $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$.*

Proof. Let $G \in \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and let $H \leq_k G$ be a k -subgroup of G . Suppose that G fits into a short exact sequence $1 \rightarrow T \rightarrow G \rightarrow F \rightarrow 1$, where $T \in \mathcal{T}_k$ and $F \in \mathcal{F}_k$. Then $H \cap T \in \mathcal{M}_k$ and $H/(T \cap H) \in \mathcal{F}_k$, so H fits into the short exact sequence

$$1 \rightarrow H \cap T \rightarrow H \rightarrow H/(H \cap T) \rightarrow 1,$$

that is, $H \in \text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$. By Lemma 5.4, we deduce that $H \in \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. An analogous proof also gives the result for $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$, once we notice that $G \in \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$ implies, since solvable groups are closed under extensions, that G is solvable, meaning that H is also solvable and thus that $H/(T \cap H) \in \mathcal{F}_k^{\text{Sol}}$. \square

Lemma 5.6. *\mathcal{M}_k is closed under k -subgroups and under “base changing/restricting”.*

Proof. The first statement is clear, by definition of groups of multiplicative type. For the second statement, let $M \in \mathcal{M}_k$ and let l/k be a finite extension. Then, since “being a group of multiplicative type” is a geometric condition, $M_l \in \mathcal{M}_l$. Moreover, since $\mathcal{M}_l = \text{Ext}(\mathcal{F}_l, \mathcal{T}_l) \cap \text{Ab}_l$ and since the Weil restriction preserves tori, commutative groups, finite groups, and short exact sequences, it follows that $\mathfrak{R}_{l/k}(M_l) \in \text{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \text{Ab}_k = \mathcal{M}_k$. \square

Lemma 5.7. *$\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under k -twists. The same is true for $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$.*

Proof. By Lemma 5.6, we can apply Lemma 2.11 with $\mathcal{B} = \mathcal{M}_k$ to conclude that $\text{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ is closed under taking k -twists. By Lemma 5.4, we have that $\text{Ext}(\mathcal{F}_k, \mathcal{M}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, hence the result. The same proof works for $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$ as well. \square

Hence, Condition 3 and its “solvable version” hold.

Since all the hypotheses in the statement of Theorem 3.1 are satisfied, by applying it we have just proved the following.

Theorem 5.8. *Let X be a nice variety over a number field k . Then*

$$X(\mathbb{A}_k)^{\text{ét}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)}.$$

Similarly,

$$X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)}.$$

Lemma 5.9. *$\text{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathcal{RR}_k$. In particular, $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \mathcal{RR}_k^{\text{Sol}}$.*

Proof. We follow [Mey10], proof of Lemma 2.2. We first prove that if $G \in \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, then $G \in \mathcal{RR}_k$; since $\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking k -subgroups, it follows that $G \in \mathcal{RR}_k$. So let $G \in \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ with short exact sequence $1 \rightarrow T \rightarrow G \rightarrow F \rightarrow 1$. Then G^0/T is trivial, since finite and connected, meaning that $T = G^0$. It follows that $R_u(G) = R_u(G^0) = R_u(T)$ is trivial. Hence, $G \in \mathcal{RR}_k$, as required.

For the converse implication, suppose that $G \in \mathcal{L}_k \setminus \text{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. Then clearly $G^0 \notin \mathcal{T}_k$. If $G^0 \notin \mathcal{RR}_k$, then we are done. So suppose that $G^0 \in \mathcal{RR}_k$. Its commutator $\mathcal{D}G^0$ is a connected linear k -subgroup, which is reductive and semisimple (since G^0 is reductive) ([Mil], Theorem

15.1), and non-trivial, since otherwise $G^0 \in \mathcal{T}_k$, a contradiction. Since $\mathcal{D}G^0$ is connected and semisimple, it contains a k -subgroup of type A_1 ([Spr98], Theorem 7.2.4). But groups of type A_1 all have non-reductive k -subgroups, so we are done.

The last claim is clear, since $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \text{Sol}_k$. \square

Corollary 5.10. *Let X be a nice variety over a number field k . Then*

$$X(\mathbb{A}_k)^{\text{ét}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{M}_k)} = X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k}.$$

Similarly,

$$X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{M}_k)} = X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k^{\text{Sol}}}.$$

Proof. Clear from Theorem 5.8, Lemma 5.4, and Lemma 5.9. \square

6. INTERLUDE

Theorem 6.1. *Let X be a nice variety over k . Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \text{Ext}(\mathcal{A}, \mathcal{U}_k)$. Then $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$.*

Before we can prove the above theorem, we need to recall some preliminary results.

Let $\mathcal{A} \subset \mathcal{B}$ be as in the statement of Theorem 6.1 and let $G \in \mathcal{B} \subset \text{Ext}(\mathcal{A}, \mathcal{U}_k)$, say fitting into the short exact sequence $1 \rightarrow U_G \rightarrow G \rightarrow A_G \rightarrow 1$, with $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. We want to show that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets, for $K = k$ and $K = k_v$, for all $v \in \Omega_k$.

Since we are dealing with pointed sets, showing that $\ker(H^1(K, G) \rightarrow H^1(K, A_G))$ is trivial is not enough to conclude injectivity. Fortunately, U_G is a normal K -subgroup of G , meaning that we can use the following result.

Proposition 6.2 (§I.5.5 Corollary 2, [Ser01]). *Let $[\tau] \in H^1(K, G)$. Then the elements of $H^1(K, G)$ with the same image in $H^1(K, A_G)$ as $[\tau]$ are in bijection with the elements of the quotient of $H^1(K, U_G^\tau)$ by the action of the group $H^0(K, A_G^\tau)$.* \square

Corollary 6.3. *The map of pointed sets $H^1(K, G) \rightarrow H^1(K, A_G)$ is injective.*

Proof. Since \mathcal{U}_K is closed under K -twists (as “being unipotent” is a geometric property), $U_G^\tau \in \mathcal{U}_K$. But $H^1(K, U^\tau)$ is trivial (cf. [Ser01], §III.2.1, Proposition 6), meaning that the quotient of $H^1(K, U_G^\tau)$ by the action of the group $H^0(K, A_G^\tau)$ is also trivial. The result then follows by Proposition 6.2. \square

The surjectivity of $H^1(K, G) \rightarrow H^1(K, A_G)$ follows from a “vanishing” theorem for unipotent K -liens by Douai (see [Bor93], Corollary 4.2), which requires some knowledge of non-abelian second Galois cohomology. We refer the reader to [FSS98] (1.2) and (5.1), and to [Bor93] for more details on the construction of K -liens and on the theory of non-abelian cohomology.

Finally, since $H^1(K, G) \rightarrow H^1(K, A_G)$ preserves base points, we have the following.

Corollary 6.4. *For $K = k$ or $K = k_v$, for any place v , we have that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets.* \square

Proof of Theorem 6.1. Let $G \in \mathcal{B}$. Since $\mathcal{B} \subset \text{Ext}(\mathcal{A}, \mathcal{U}_k)$, G fits into a short exact sequence

$$1 \rightarrow U_G \rightarrow G \rightarrow A_G \rightarrow 1,$$

where $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. Fix $[g : Y \rightarrow X] \in H^1(X, G)$. We can push forward Y along $p : G \rightarrow A_G$ to obtain $[a_g : Z \rightarrow X] \in H^1(X, A_G)$.

The inclusion $X(\mathbb{A}_k)^g \subset X(\mathbb{A}_k)^{ag}$ is easy. For the opposite inclusion, let $(x_v) \in X(\mathbb{A}_k)^{ag}$, so that $[Z]((x_v)) \in \text{Im}(H^1(k, A_G) \rightarrow \prod_{v \in \Omega_k} H^1(k_v, A_G))$. Our aim is to show that $[Y]((x_v)) \in \text{Im}(H^1(k, G) \rightarrow \prod_{v \in \Omega_k} H^1(k_v, G))$, as, using the functorial description of obstructions, this would indeed imply that $(x_v) \in X(\mathbb{A}_k)^g$.

By Corollary 6.4, for any field K containing k ,

$$(6.1) \quad p_* : H^1(K, G) \xrightarrow{\sim} H^1(K, A_G)$$

is an isomorphism of pointed sets, where p_* is induced by the projection $p : G \rightarrow A_G$. In particular, (6.1) holds for $K = k$ or $K = k_v$, for any place $v \in \Omega_k$.

Now, for each place $v \in \Omega_k$, we have the commutative diagram

$$\begin{array}{ccccc} H^1(X, G) & \xrightarrow{x_v^*} & H^1(k_v, G) & \xleftarrow{\text{res}_v} & H^1(k, G) \\ \downarrow p_*^X & & p_*^{-1} \updownarrow p_* & & p_*^{-1} \updownarrow p_* \\ H^1(X, A_G) & \xrightarrow{x_v^*} & H^1(k_v, A_G) & \xleftarrow{\text{res}_v} & H^1(k, A_G). \end{array}$$

From the left square, we deduce that

$$p_*([Y](x_v)) = p_*^X[Y](x_v) = [Z](x_v) \in \text{Im}(H^1(k, A_G) \rightarrow H^1(k_v, A_G)).$$

But $H^1(k, A_G) = p_*(H^1(k, G))$ and $H^1(k_v, A_G) = p_*(H^1(k_v, G))$, by (6.1). This fact and the commutativity of the right square in the above diagram together imply that

$$p_*([Y](x_v)) \in \text{Im}(p_*(H^1(k, G)) \rightarrow p_*(H^1(k_v, G))) = p_*(\text{Im}(H^1(k, G) \rightarrow H^1(k_v, G))).$$

By applying p_*^{-1} , we get $[Y](x_v) \in \text{Im}(H^1(k, G) \rightarrow H^1(k_v, G))$; taking the product over all places $v \in \Omega_k$ then gives the required result. Hence, $X(\mathbb{A}_k)^{ag} \subset X(\mathbb{A}_k)^g$, and so

$$(6.2) \quad X(\mathbb{A}_k)^{ag} = X(\mathbb{A}_k)^g.$$

Intersecting over all torsors in $H^1(X, G)$ and all $G \in \mathcal{B}$ gives

$$(6.3) \quad \bigcap_{G \in \mathcal{B}} \bigcap_{[Y] \in H^1(X, G)} X(\mathbb{A}_k)^{ag} = \bigcap_{G \in \mathcal{B}} \bigcap_{[Y] \in H^1(X, G)} X(\mathbb{A}_k)^g.$$

For any $G \in \mathcal{A} \subset \mathcal{B}$, we have $G = A_G$ and $g = a_g$. Hence, the left-hand side of (6.3) can be rewritten as

$$\bigcap_{G \in \mathcal{B}} \bigcap_{[g: Y \rightarrow X] \in H^1(X, G)} X(\mathbb{A}_k)^{ag} = X(\mathbb{A}_k)^{\mathcal{A}} \cap \underbrace{\left(\bigcap_{G \in \mathcal{B} \setminus \mathcal{A}} \bigcap_{[a_g: Z \rightarrow X] \in H^1(X, A_G)} X(\mathbb{A}_k)^{ag} \right)}_{\supset X(\mathbb{A}_k)^{\mathcal{A}}} = X(\mathbb{A}_k)^{\mathcal{A}}.$$

Hence, since the right-hand side of (6.3) is, by definition, $X(\mathbb{A}_k)^{\mathcal{B}}$, we get $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$, as required. \square

Remark 6.5. In Theorem 6.1, we use \mathcal{U}_k just to guarantee that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets for $K = k$ and $K = k_v$, for all $v \in \Omega_k$ (cf. (6.1)). In theory, we can get a similar result as Theorem 6.1 if we replace \mathcal{U}_k with any other $\mathcal{S} \subset \mathcal{L}_k$ such that (6.1) holds for $K = k$ and $K = k_v$, for all $v \in \Omega_k$: for example, when k is a totally imaginary number field, a good candidate for such an \mathcal{S} is the set of semisimple simply connected linear algebraic groups over k – see §3.1 of [Ser01] and Corollary 5.1 of [Bor93].

Example 6.6. Any $G \in \mathcal{A}b_k$ can be written as $G = U \times M$, where $U \in \mathcal{U}_k$ and $M \in \mathcal{M}_k$ (cf. [Mil12], 1.21). Hence, we have the inclusions $\mathcal{M}_k \subset \mathcal{A}b_k \subset \text{Ext}(\mathcal{M}_k, \mathcal{U}_k)$, implying that $X(\mathbb{A}_k)^{\mathcal{M}_k} = X(\mathbb{A}_k)^{\mathcal{A}b_k}$.

Corollary 6.7. *Let $G \in \mathcal{L}_k$. Then $X(\mathbb{A}_k)^{H^1(X, G)} = X(\mathbb{A}_k)^{H^1(X, R_G)}$, where $R_G := G/R_u(G)$.*

Proof. Let $\mathcal{A} = \{R_G\}$ and $\mathcal{B} = \{G, R_G\}$ in Theorem 6.1. \square

Corollary 6.8. *Let X be a nice variety over k . Then $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)$.*

Proof. Let $\mathcal{A} = \{1\}$ and $\mathcal{B} = \mathcal{U}_k$. Clearly $\mathcal{U}_k \subset \text{Ext}(\{1\}, \mathcal{U}_k) = \mathcal{U}_k$, and so the hypotheses of Theorem 6.1 are satisfied. Therefore, we can conclude that $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)^{\{1\}} = X(\mathbb{A}_k)$, as required. \square

Remark 6.9. Since $H^1(K, U)$ is trivial for all $U \in \mathcal{U}_k$, for $K = k$ and $K = k_v$, for all $v \in \Omega_k$ (cf. [Ser01], §III.2.1, Proposition 6), we have that, in the proof of Corollary 6.8, (6.1) holds without having to resort to non-abelian second Galois cohomology.

Since linear algebraic unipotent groups over k give no obstruction, and since reductive groups are, in a sense, the “opposite” of unipotent groups, the following shouldn’t be too surprising.

Corollary 6.10. *Let X be a nice variety over k . Then $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{R}_k$ and $\mathcal{B} = \mathcal{L}_k$. It is clear that $\mathcal{L}_k = \text{Ext}(\mathcal{R}_k, R_u(\mathcal{L}_k)) \subset \text{Ext}(\mathcal{R}_k, \mathcal{U}_k)$. This means that the hypotheses of Theorem 6.1 are satisfied. Therefore, we can conclude that $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$, as required. \square

6.1. Some corollaries. The above also suggests the following proposition.

Proposition 6.11. *Let $\mathcal{B} \subset \mathcal{L}_k$ be such that $B/R_u(B) \in \mathcal{B}$, for any $B \in \mathcal{B}$. Let X be a nice variety over k . Then $X(\mathbb{A}_k)^{\mathcal{B}} = X(\mathbb{A}_k)^{\mathcal{B} \cap \mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{B} \cap \mathcal{R}_k \subset \mathcal{B}$. By assumption $B/R_u(B) \in \mathcal{B} \cap \mathcal{R}_k$ for any $B \in \mathcal{B}$, so we can apply Theorem 6.1 to conclude the result. \square

Note that, for example, $\mathcal{B} = \mathcal{R}_k, \mathcal{C}_k, \mathcal{A}b_k, \text{Sol}_k$ all satisfy the hypotheses of Proposition 6.11, as they are all closed under taking quotients. Hence, we can immediately deduce the following.

Corollary 6.12. $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\mathcal{C}_k \cap \mathcal{R}_k}$. \square

Remark 6.13. The above could have also been deduced in the following way. Let $\text{PGL} := \cup_{n \geq 1} \text{PGL}_n$. For X nice over k , we know that $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)^{\text{PGL}}$: the last equality is Proposition 5.3.4 in [Sko01], while the first one follows from [Har02]. Since $\mathcal{R}_k \cap \mathcal{C}_k \subset \mathcal{C}_k$, it follows immediately that $X(\mathbb{A}_k)^{\mathcal{C}_k} \subset X(\mathbb{A}_k)^{\mathcal{C}_k \cap \mathcal{R}_k}$. Now, for any $n \geq 1$, PGL_n is both connected and reductive. It follows that $\text{PGL} \subset \mathcal{R}_k \cap \mathcal{C}_k$, meaning that $X(\mathbb{A}_k)^{\mathcal{C}_k \cap \mathcal{R}_k} \subset X(\mathbb{A}_k)^{\text{PGL}} = X(\mathbb{A}_k)^{\mathcal{C}_k}$. Hence, $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\mathcal{C}_k \cap \mathcal{R}_k}$.

Corollary 6.14. $X(\mathbb{A}_k)^{\text{Sol}_k} = X(\mathbb{A}_k)^{\text{Sol}_k \cap \mathcal{R}_k}$. In particular, $X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Sol}_k}$.

Proof. By Proposition 6.11, the first statement is obvious, so we just need to prove the last statement. We show that $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Sol}_k \cap \mathcal{R}_k$. Let $G \in \text{Sol}_k \cap \mathcal{R}_k$. Then G^0 is reductive, connected, and solvable. Hence, $G^0 \in \mathcal{T}_k$. Moreover, since $G \in \text{Sol}_k$, then $G/G^0 \in \mathcal{F}_k^{\text{Sol}}$. Conversely, let $G \in \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k)$. Then $G \in \text{Sol}_k$, as solvable groups are closed under extensions, and $G \in \mathcal{R}_k$ by Lemma 5.9. Hence, $\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{T}_k) = \text{Sol}_k \cap \mathcal{R}_k$. From Corollary 5.10 and the fact that $X(\mathbb{A}_k)^{\text{Sol}_k} = X(\mathbb{A}_k)^{\text{Sol}_k \cap \mathcal{R}_k}$, we can then deduce the required result. \square

Corollary 6.15. $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k) \cap \mathcal{R}_k}$.

Proof. Let $G \in \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)$. We need to check that $G/R_u(G) \in \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)$. We know that G fits into a short exact sequence

$$1 \rightarrow C \rightarrow G \rightarrow F \rightarrow 1,$$

where $C \in \mathcal{C}_k$ and $F \in \mathcal{F}_k^{\text{Sol}}$. Notice that $C = G^0$. Indeed, by definition of G^0 , we have $C \subset G^0$, and since C is normal in G , it follows that C is also normal in G^0 . Consider the short exact sequence $1 \rightarrow G^0 \rightarrow G \rightarrow G/G^0 \rightarrow 1$, where $G/G^0 \in \mathcal{F}_k$. This induces the short exact sequence

$$1 \rightarrow G^0/C \rightarrow G/C \cong F \rightarrow G/G^0 \rightarrow 1;$$

since G^0/C is connected and injects into the finite group F , it must be trivial, i.e. $C = G^0$. Hence, G fits into the short exact sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow F \rightarrow 1.$$

Now, since $R_u(G)$ is, by definition, a connected normal subgroup of G , we have that $R_u(G)$ is normal in G^0 . Hence, we get the short exact sequence

$$1 \rightarrow G^0/R_u(G) \rightarrow G/R_u(G) \rightarrow F \rightarrow 1,$$

where $G^0/R_u(G) \in \mathcal{C}_k$ and $F \in \mathcal{F}_k^{\text{Sol}}$, meaning that $G/R_u(G) \in \text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)$. By Proposition 6.11, the result follows. \square

Remark 6.16. A similar proof also shows that $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Ab}}, \mathcal{C}_k)} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Ab}}, \mathcal{C}_k) \cap \mathcal{R}_k}$, and similarly for any subset of \mathcal{F}_k extended by \mathcal{C}_k .

Corollary 6.17. $X(\mathbb{A}_k)^{\mathcal{T}_k} = X(\mathbb{A}_k)^{\text{Ab}_k \cap \mathcal{C}_k} = X(\mathbb{A}_k)^{\text{Sol}_k \cap \mathcal{C}_k}$.

Proof. Since both Sol_k and \mathcal{C}_k are closed under quotients, so is their intersection, meaning that we can apply Proposition 6.11 to deduce that $X(\mathbb{A}_k)^{\text{Sol}_k \cap \mathcal{C}_k} = X(\mathbb{A}_k)^{\text{Sol}_k \cap \mathcal{C}_k \cap \mathcal{R}_k} = X(\mathbb{A}_k)^{\mathcal{T}_k}$. Similarly, by noticing that $\mathcal{T}_k = \mathcal{C}_k \cap \mathcal{R}_k \cap \text{Ab}_k$ and that abelian groups are also closed under quotients, we get the other equality in the statement of the corollary. \square

7. A NETWORK OF OBSTRUCTIONS

7.1. Summary. For any nice variety X over k , Figure 1 summarises the relations between some of the obstruction sets mentioned in this paper. The rest of this section is to remind the reader of why all the inclusions in Figure 1 can be strict.

Proposition 7.1. *In general, $X(k) \subsetneq X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét}, \text{Br}}$, for X a nice variety over k .*

Proof. This is the content of Poonen's paper [Poo10]. \square

Proposition 7.2. *In general, $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\text{Ab}_k/\mathcal{M}_k/\text{Br}_1} \subsetneq X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k}$, for X a nice variety over k .*

Proof. See [BSD75] for an example over $k = \mathbb{Q}$. \square

Remark 7.3. More strongly, we have that, in general, $X(\mathbb{A}_k)^{\mathcal{T}_k/\mathcal{C}_k^{\text{Ab}}/\mathcal{C}_k^{\text{Sol}}} \subsetneq X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k}$. Indeed, let X be the (nice) degree 4 del Pezzo surface over $k = \mathbb{Q}$ in [BSD75]. We know that $X(\mathbb{A}_k)^{\text{Br}_1} = \emptyset$ but $X(\mathbb{A}_k) \neq \emptyset$. Moreover, since X is geometrically rational, $\pi_1(\overline{X}) = 0$, meaning that there are no non-trivial finite torsors over X . In particular, $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{T}_k)} = X(\mathbb{A}_k)^{\mathcal{T}_k}$. Hence, $\emptyset = X(\mathbb{A}_k)^{\text{Br}_1} = X(\mathbb{A}_k)^{\mathcal{M}_k} = X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \text{Ab}_k} = X(\mathbb{A}_k)^{\mathcal{T}_k} \neq X(\mathbb{A}_k)$.

$$\begin{array}{ccccc}
 & & & & X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k} \\
 & & & & \uparrow \\
 & & & & X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Ab}}} \\
 & & & & \uparrow \\
 & & & & X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} \\
 & & & & \uparrow \\
 & & & & X(\mathbb{A}_k)^{\mathcal{F}_k} \\
 & & & & \uparrow \\
 & & & & X(k) \\
 & & & & \uparrow \\
 X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\text{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\text{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k} \\
 \uparrow & & \uparrow & & \uparrow \\
 X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, C_k)/\text{Ext}(\mathcal{F}_k^{\text{Sol}}, C_k) \cap \mathcal{R}_k/\text{Sol},\text{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k^{\text{Sol}}/\mathcal{R}_k^{\text{Sol}}/\text{Sol}_k/\text{Sol},\text{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} \\
 \uparrow & & \uparrow & & \uparrow \\
 X(\mathbb{A}_k)^{\mathcal{C}_k/C_k \cap \mathcal{R}_k/\text{PGL}/\text{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\text{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Ab}}} \\
 \uparrow & & \uparrow & & \uparrow \\
 & & & & X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k}
 \end{array}$$

FIGURE 1.

Proposition 7.4. *In general, for X a nice variety over k ,*

- (a) $X(\mathbb{A}_k)^{\text{Ab},\text{Br}_1} \subsetneq X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\text{Br}_1}$;
- (b) $X(\mathbb{A}_k)^{\text{Ab},\text{Br}} \subsetneq X(\mathbb{A}_k)^{\mathcal{C}_k/C_k \cap \mathcal{R}_k/\text{PGL}/\text{Br}}$;
- (c) *the column in Figure 1 with endpoints $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\text{Br}_1}$ and $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\text{Br}_1}$ can be strictly contained in the column with endpoints $X(\mathbb{A}_k)^{\mathcal{F}_k}$ and $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Ab}}}$;*
- (d) $X(\mathbb{A}_k)^{\mathcal{C}_k/C_k \cap \mathcal{R}_k/\text{PGL}/\text{Br}} \not\subset X(\mathbb{A}_k)^{\text{Ab},\text{Br}_1}$;
- (e) $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} \not\subset X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\text{Br}_1}$.

Proof. Let $Y = Y_{a,b,c}$ and $X = Y_{a,b,c}/\langle \iota \rangle$ be, respectively, the (nice) K3 and Enriques surfaces over $k = \mathbb{Q}$ from [VAV11]. Then $(f, Y, F) \in \mathcal{F}_k^{\text{Ab}}(X)$, where $F = \mathbb{Z}/2\mathbb{Z}$.

(a), (b), (d). Using Kummer theory and following [VAV11], it is not difficult to show that $\bigcup_{[\tau] \in H^1(X, F)} f^\tau(Y^\tau(\mathbb{A}_k)^{\text{Br}_1}) = \emptyset$, meaning, in particular, that $X(\mathbb{A}_k)^{\text{Ab},\text{Br}_1} = \emptyset$. Moreover, in [BBM⁺], the authors have shown that $X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$.

(c), (e). Note that $Y_{a,b,c}(\mathbb{A}_k)^{\mathcal{F}_k} = Y_{a,b,c}(\mathbb{A}_k) \neq \emptyset$ and $Y_{a,b,c}(\mathbb{A}_k)^{\text{ét},\text{Br}_1} = Y_{a,b,c}(\mathbb{A}_k)^{\text{Br}_1} = \emptyset$. \square

Proposition 7.5. *In general, for X a nice variety over k ,*

- (a) $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} \subsetneq X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Ab}}}$;
- (b) $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\text{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\text{Br}_1} \not\subset X(\mathbb{A}_k)^{\mathcal{F}_k}$.

Proof. (a), (b). Let X be the (nice) bielliptic surface over $k = \mathbb{Q}$ from Skorobogatov's counterexample (see [Sko01], Chapter 8). Then $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} = \emptyset$ (cf. §5.1 of [HS02]), but $X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$. \square

Proposition 7.6. *In general, for X a nice variety over k ,*

- (a) *the column in Figure 1 starting with $X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\text{Br}}$ can be strictly contained in the column starting with $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\text{Br}_1}$;*
- (b) $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\text{Br}_1} \not\subset X(\mathbb{A}_k)^{\mathcal{C}_k/C_k \cap \mathcal{R}_k/\text{PGL}/\text{Br}}$.

Proof. (a), (b). In [HVA13], the authors have constructed a (nice) K3 surface X over $k = \mathbb{Q}$ such that $X(\mathbb{A}_k)^{\text{Br}} = \emptyset$ but $X(\mathbb{A}_k)^{\text{Br}_1} \neq \emptyset$. Now, since any K3 surface is simply connected, we have that $\pi_1(\bar{X}) = 0$. In particular, we have that $X(\mathbb{A}_k)^{\text{ét},\text{Br}} = X(\mathbb{A}_k)^{\text{Br}}$ and $X(\mathbb{A}_k)^{\text{ét},\text{Br}_1} = X(\mathbb{A}_k)^{\text{Br}_1}$. \square

Proposition 7.7. *In general, the row in Figure 1 starting with $X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\acute{e}t, \text{Br}}$ can be strictly contained in the row starting with $X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k)/\text{Ext}(\mathcal{F}_k^{\text{Sol}}, \mathcal{C}_k) \cap \mathcal{R}_k/\text{Sol}, \text{Br}}$, for X a nice variety over k .*

Proof. Let G be a finite, perfect (i.e. with $G = \mathcal{D}G$), non-abelian, simple group; in particular, G is not solvable. Let X and k be as in the conclusion of Corollary 6.1 in [Har00]. Since $G = \pi(\overline{X})$ and G is simple, it follows that $X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} = X(\mathbb{A}_k)$, as there are no non-trivial finite solvable covers of X . This implies that $X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Br}_1}$ and $X(\mathbb{A}_k)^{\text{Sol}, \text{Br}} = X(\mathbb{A}_k)^{\text{Br}}$. Since G is perfect, $\text{Br}(\overline{X}) = 0$ (cf. the remark after Corollary 6.1 in [Har00]), meaning that $X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)^{\text{Br}_1}$ and that $\text{Br}(X)/\text{Br}k$ is finite. By taking k large enough, we can assume that $\text{Br}(X)/\text{Br}k$ is trivial and, thus, that $X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)$. Hence,

$$X(\mathbb{A}_k)^{\text{Sol}, \text{Br}_1} = X(\mathbb{A}_k)^{\text{Br}_1} = X(\mathbb{A}_k)^{\text{Sol}, \text{Br}} = X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)^{\mathcal{F}_k^{\text{Sol}}} = X(\mathbb{A}_k).$$

But, by construction of X , we have $X(\mathbb{A}_k)^{\mathcal{F}_k} \subsetneq X(\mathbb{A}_k)$. Since $X(\mathbb{A}_k)^{\acute{e}t, \text{Br}} \subset X(\mathbb{A}_k)^{\acute{e}t, \text{Br}_1} \subset X(\mathbb{A}_k)^{\mathcal{F}_k}$, the result follows. \square

We conclude with an open question.

Question 7.8. What are, if they exist, the translations of $X(\mathbb{A}_k)^{\text{Ab}, \text{Br}}$ and $X(\mathbb{A}_k)^{\text{Ab}, \text{Br}_1}$ in pure “descent-type” terms?

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