ITERATING THE ALGEBRAIC ÉTALE-BRAUER SET

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ABSTRACT. In this paper, we iterate the algebraic étale-Brauer set for any nice variety X over a number field k with $\pi_1^{\text{ét}}(\overline{X})$ finite and we show that the iterated set coincides with the original algebraic étale-Brauer set. This provides some evidence towards the conjectures by Colliot-Thélène on the arithmetic of rational points on nice geometrically rationally connected varieties over k and by Skorobogatov on the arithmetic of rational points on K3 surfaces over k; moreover, it gives a partial answer to an "algebraic" analogue of a question by Poonen about iterating the descent set.

1. INTRODUCTION

1.1. **Notation.** In this paper, $k \subset \mathbf{C}$ will be a number field, $\overline{k} \subset \mathbf{C}$ a fixed algebraic closure of k, $\mathbf{A}_k$ the ring of adeles of k, Ω_k the set of places of k, and k_v the completion of k at $v \in \Omega_k$. For any variety X over k, we endow $X(\mathbf{A}_k)$ with the adelic topology and $\prod_{v\in\Omega_k} X(k_v)$ with the product topology; when X is proper, $X(\mathbf{A}_k) = \prod_{v\in\Omega_k} X(k_v)$ and the product and adelic topologies are equivalent. A variety which is smooth, projective, and geometrically integral over k will be called a nice variety over k. Let $\{X_\omega\}_\omega$ be a family of smooth, geometrically integral varieties over k. If $X(\mathbf{A}_k) \neq \emptyset \iff X(k) \neq \emptyset$ for all $X \in \{X_\omega\}_\omega$, we say that $\{X_\omega\}_\omega$ satisfies the Hasse principle (HP), while if $X(k) \neq \emptyset$ and $\overline{X(k)} = X(\mathbf{A}_k)$ for all $X \in \{X_\omega\}_\omega$, we say that $\{X_\omega\}_\omega$ satisfies strong approximation (SA). When the varieties in the family $\{X_\omega\}_\omega$ are moreover proper, strong approximation is equivalent to weak approximation (WA), i.e. to the property that $X(k) \neq \emptyset$ and $\overline{X(k)} = \prod_{v\in\Omega_k} X(k_v)$ for all $X \in \{X_\omega\}_\omega$; in general, however, we just have the chain of implications (SA) $\Rightarrow$ (WA) $\Rightarrow$ (HP). For any smooth variety X over k, the Brauer group of X is $\operatorname{Br} X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ and the Brauer-Manin set of X is

$$X(\mathbf{A}_k)^{\operatorname{Br}} := \bigcap_{\alpha\in\operatorname{Br} X} \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v\in\Omega_k} \operatorname{inv}_v(\alpha(x_v)) = 0 \right\},$$

where the $\operatorname{inv}_v : \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ are the local invariant maps coming from class field theory. The algebraic Brauer group of X is $\operatorname{Br}_1(X) := \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$, where $\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ and where $\operatorname{Br} X \to \operatorname{Br} \overline{X}$ is the canonical map induced by the natural morphism $\overline{X} \to X$. We define the algebraic Brauer-Manin set $X(\mathbf{A}_k)^{\operatorname{Br}_1}$ by restricting the intersection in the definition of the Brauer-Manin set to the elements in $\operatorname{Br}_1 X$. One can show that both $X(\mathbf{A}_k)^{\operatorname{Br}}$ and $X(\mathbf{A}_k)^{\operatorname{Br}_1}$ are closed in $X(\mathbf{A}_k)$ and that $\overline{X(k)} \subset X(\mathbf{A}_k)^{\operatorname{Br}} \subset X(\mathbf{A}_k)^{\operatorname{Br}_1}$ (see e.g. [Sko01, §5.2]).

Let $\mathcal{L}_k := \{G : G \text{ is a linear algebraic } k\text{-group}\}/\sim$, where $G_1 \sim G_2$ if and only if G_1 and G_2 are k-isomorphic as k-groups. We will abuse notation and write $G \in \mathcal{L}_k$ also to mean a representative of the k-isomorphism class of G. For any $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_k$, we let

$$\operatorname{Ext}(\mathcal{A}, \mathcal{B}) = \{G \in \mathcal{L}_k : G \text{ is an extension of } A \text{ by } B, \text{ for some } A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}/\sim .$$

For any $\mathcal{S} \subset \mathcal{L}_k$, the $\mathcal{S}$-descent set is

$$X(\mathbf{A}_k)^{\mathcal{S}} := \bigcap_{G\in\mathcal{S}} \ \bigcap_{[f:Y\to X]\in H^1_{\text{ét}}(X,G)} \ \bigcup_{[\tau]\in H^1_{\text{ét}}(k,G)} f^\tau(Y^\tau(\mathbf{A}_k));$$

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when $\mathcal{S}=\emptyset$, we define $X(\mathbf{A}_k)^{\emptyset}:=X(\mathbf{A}_k)$, while when $\mathcal{S}=\mathcal{L}_k$, the $\mathcal{L}_k$-descent set is just called descent set. For any $\mathcal{S}\subset\mathcal{L}_k$, the set $X(\mathbf{A}_k)^{\mathcal{S}}$ is closed in $X(\mathbf{A}_k)$ and contains the adelic closure of $X(k)$ (see [CDX16, Prop. 6.4]). Let $\mathcal{F}_k:=\{G\in\mathcal{L}_k : G \text{ is finite}\}/\sim$.

The étale-Brauer set of X is

$$X(\mathbf{A}_k)^{\text{ét Br}}:=\bigcap_{F\in\mathcal{F}_k}\ \bigcap_{[f:Y\to X]\in H^1_{\text{ét}}(X,F)}\ \bigcup_{[\tau]\in H^1_{\text{ét}}(k,F)} f^{\tau}(Y^{\tau}(\mathbf{A}_k)^{\mathrm{Br}}).$$

Similarly, we can define the algebraic étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}_1}$ by replacing "Br" with "Br_1" in the definition above. Both $X(\mathbf{A}_k)^{\text{ét Br}}$ and $X(\mathbf{A}_k)^{\text{ét Br}_1}$ are closed in $X(\mathbf{A}_k)$ (see the discussion after [CDX16, Prop. 6.6]), and they both contain the adelic closure of $X(k)$. Finally, for any $\mathcal{S},\mathcal{S}'\subset\mathcal{L}_k$ and any $\star\in\{\emptyset,\mathrm{Br},\mathrm{Br}_1,\text{ét Br},\text{ét Br}_1,\mathcal{S}'\}$, we define

$$\mathrm{Iter}_{\mathcal{S}}(X_{/k},\star):=\bigcap_{G\in\mathcal{S}}\ \bigcap_{[f:Y\to X]\in H^1_{\text{ét}}(X,G)}\ \bigcup_{[\tau]\in H^1_{\text{ét}}(k,G)} f^{\tau}(Y^{\tau}(\mathbf{A}_k)^{\star}).$$

1.2. **Motivation.** The aim of this paper is to give some evidence and partial answers to various conjectures and open questions about the arithmetic behaviour of rational points on certain classes of varieties over k. More specifically, the conjectures that we are interested in are the following.

Conjecture 1.1 (Colliot-Thélène, [CT03, p.174])**.** Let X be a nice geometrically rationally connected variety over k. Then $\overline{X(k)}=X(\mathbf{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction is the only one for strong (equivalently, weak) approximation.

(Recall that X is geometrically rationally connected if any two general points $x_1,x_2\in\overline{X}$ can be joined by a chain of $\overline{k}$-rational curves; examples of geometrically rationally connected varieties include geometrically unirational varieties and Fano varieties.)

Conjecture 1.2 (Skorobogatov)**.** Let X be a nice K3 surface over k. Then $\overline{X(k)}=X(\mathbf{A}_k)^{\mathrm{Br}}$. In other words, the Brauer-Manin obstruction is the only one for strong (equivalently, weak) approximation.

Conjecture 1.3. Let X be a nice Enriques surface over k. Then $\overline{X(k)}=X(\mathbf{A}_k)^{\text{ét Br}}$. In other words, the étale-Brauer obstruction is the only one for strong (equivalently, weak) approximation.

Remark 1.4. In general, K3 surfaces over k do not satisfy $\overline{X(k)}=X(\mathbf{A}_k)$; see [HVA13] for an example over $k=\mathbf{Q}$ violating the Hasse principle. Similarly, in [BBM$^+$16], the authors have constructed an Enriques surface X over $k=\mathbf{Q}$ such that $X(\mathbf{A}_k)^{\mathrm{Br}}\neq\emptyset$ but $X(k)=\emptyset$; this implies that, for Enriques surfaces, $\overline{X(k)}=X(\mathbf{A}_k)^{\mathrm{Br}}$ does not hold in general.

Another source of motivation for this paper is the following: for any nice variety X over k, the étale-Brauer set $X(\mathbf{A}_k)^{\text{ét Br}}$ is currently the smallest general obstruction set known. Unfortunately, the étale-Brauer set is not small enough to explain all the failures of the Hasse principle: see e.g. [Poo10] for a counterexample. We thus want a way to construct obstruction sets smaller than $X(\mathbf{A}_k)^{\text{ét Br}}$. A possible strategy is to mimick the construction of the étale-Brauer set itself: for any nice variety X over k, the results in [Dem09b] and [Sko09] imply that $X(\mathbf{A}_k)^{\text{ét Br}}=X(\mathbf{A}_k)^{\mathcal{L}_k}$; if $\mathcal{S}\subset\mathcal{L}_k$ contains the trivial group, then the obstruction set $\mathrm{Iter}_{\mathcal{S}}(X_{/k},\mathcal{L}_k)$ is certainly *potentially* smaller than $X(\mathbf{A}_k)^{\text{ét Br}}$. It turns out, however, that for certain choices of $\mathcal{S}$ the set $\mathrm{Iter}_{\mathcal{S}}(X_{/k},\mathcal{L}_k)$ is the same as the original étale-Brauer set: this is the case, for example, when $\mathcal{S}=\mathcal{F}_k$ (cf. [Sko09, Thm 1.1]). It is natural to ask about the case when $\mathcal{S}$ is maximal, i.e. when $\mathcal{S}=\mathcal{L}_k$; in this case, we can think of $\mathrm{Iter}_{\mathcal{L}_k}(X_{/k},\mathcal{L}_k)$ as an "iteration" of the descent set.

Question 1.5 (Poonen)**.** Let X be a nice variety over k. Is $\mathrm{Iter}_{\mathcal{L}_k}(X_{/k},\mathcal{L}_k)=X(\mathbf{A}_k)^{\text{ét Br}}$?

Remark 1.6. In [CDX16, Thm 7.5], the authors show that $Y(\mathbf{A}_k)^{\mathcal{L}_k}=Y(\mathbf{A}_k)^{\text{ét Br}}$ for any smooth, quasi-projective, geometrically connected variety Y over k, thus removing the properness condition from the earlier results in [Dem09b] and [Sko09]. As a consequence, we have that $\mathrm{Iter}_{\mathcal{L}_k}(X_{/k}, \mathcal{L}_k) = \mathrm{Iter}_{\mathcal{L}_k}(X_{/k}, \text{ét}\,\mathrm{Br})$ for any nice variety X over k.

We focus on a question similar to Question 1.5: we want to iterate the algebraic étale-Brauer set $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$. To make sense of this, we first need an analogue of the result $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}} = X(\mathbf{A}_k)^{\mathcal{L}_k}$ for $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$. Such an analogue is given by [Bal16, Thm 5.8]: if X is a nice variety over k, then

$$X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)},$$

where $\mathcal{T}_k := \{G \in \mathcal{L}_k : G \text{ is a torus}\}/\sim$.

Question 1.7. Let X be a nice variety over k. Is $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \text{ét}\,\mathrm{Br}_1) = X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$?

Remark 1.8. By putting together the results in [CDX16] and [Bal16], we have $Y(\mathbf{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)} = Y(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$ for any smooth, quasi-projective, geometrically connected variety Y over k. From this, we can easily deduce that $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)) = \mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \text{ét}\,\mathrm{Br}_1)$ for any nice variety X over k.

1.3. **Main result.** Motivated by the above conjectures and questions, our main theorem is the following.

Theorem 1.9 (Main Theorem)**.** *Let X be a nice variety over k such that $\pi_1^{\text{ét}}(\overline{X})$ is finite. Then* $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \text{ét}\,\mathrm{Br}_1) = X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$.

Some comments:

- Let X be a nice geometrically rationally connected variety over k. Then $\pi_1^{\text{ét}}(\overline{X}) = 0$, meaning that the hypotheses of Theorem 1.9 are satisfied. If moreover $\mathrm{Br}\,\overline{X} = 0$, as is the case when $\dim X = 2$ or $H^3_{\text{ét}}(\overline{X}, \mathbf{Z}_\ell(1))_{\mathrm{tors}} = 0$ for all primes ℓ (cf. [CTS13, Lemma 1.3]), then $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathrm{Br}_1} = X(\mathbf{A}_k)^{\mathrm{Br}}$. Hence, in this case, Theorem 1.9 tells us that $\mathrm{Iter}_{\mathcal{T}_k}(X_{/k}, \text{ét}\,\mathrm{Br}_1) = X(\mathbf{A}_k)^{\mathrm{Br}}$, thus giving some evidence for Conjecture 1.1.
- Let X be a nice K3 surface over k. Then $\pi_1^{\text{ét}}(\overline{X}) = 0$, and thus the hypotheses of Theorem 1.9 are satisfied. When, moreover, $\mathrm{Br}\,\overline{X} = 0$ (see e.g. the comment after [SZ12, Prop. 5.1]), then Theorem 1.9 yields $\mathrm{Iter}_{\mathcal{T}_k}(X_{/k}, \text{ét}\,\mathrm{Br}_1) = X(\mathbf{A}_k)^{\mathrm{Br}}$, thus giving some evidence for Conjecture 1.2.
- Let X be a nice Enriques surface over k. Then $\pi_1^{\text{ét}}(\overline{X}) \cong \mathbf{Z}/2\mathbf{Z}$, and so the hypotheses of Theorem 1.9 are satisfied. When $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}} = X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$ (e.g. in [VAV11]), then $\mathrm{Iter}_{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \text{ét}\,\mathrm{Br}_1) = X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}}$, which is some evidence for Conjecture 1.3. Theorem 1.9 also applies to some higher-dimensional analogues of Enriques surfaces (cf. [BNWS11, §2]).
- Theorem 1.9 gives a positive answer to Question 1.7, assuming the finiteness of $\pi_1^{\text{ét}}(\overline{X})$; it would be interesting to see whether this condition can be weakened or removed (a possible weakening could be that of considering nice varieties X over k such that $\mathrm{Pic}\,\overline{Y}$ is finitely generated as a $\mathbf{Z}$-module for any finite cover $Y \to X$).
- In the literature, there are several conditional proofs of the existence of nice varieties X over k with $\pi_1^{\text{ét}}(\overline{X}) = 0$, $X(k) = \emptyset$, and $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$: see [SW95] for an example conditional on Lang's conjectures, [Poo01] for one conditional on the existence of a complete intersection satisfying certain properties, and [Sme14, Thm 4.1] for one conditional on the *abc* conjecture. Theorem 1.9 would apply to these examples.

2. Some properties of universal torsors

Let X be a variety over k with $\overline{k}[X]^\times = \overline{k}^\times$ and with $\mathrm{Pic}\,\overline{X}$ finitely generated as a $\mathbf{Z}$-module. Let $\mathcal{M}_k := \{G \in \mathcal{L}_k : G \text{ is of multiplicative type}\}/\sim$ and let $S \in \mathcal{M}_k$. An S-torsor $Y \to X$ is a universal torsor for X if its type $\lambda_Y : \widehat{S} \to \mathrm{Pic}\,\overline{X}$ is an isomorphism (for the definition of the type of a torsor, see e.g. [Sko01, Cor. 2.3.9]); here $\widehat{S} := \mathrm{Hom}_{\mathbf{GrpSch}_{\overline{k}}}(\overline{S}, \mathbf{G}_{m,\overline{k}})$ denotes the Cartier dual. As explained in [Sko01, §2.3], universal torsors do not always exist over k, as a universal torsor over $\overline{k}$ might not descend to k. The following proposition gives sufficient conditions for the existence of universal torsors.

Proposition 2.1 ([Sko01, Cor. 6.1.3(1)]). *Let X be a variety over k such that $\overline{k}[X]^\times = \overline{k}^\times$, $\mathrm{Pic}\,\overline{X}$ is finitely generated as a $\mathbf{Z}$-module, and $X(\mathbf{A}_k)^{\mathrm{Br}_1} \neq \emptyset$. Then there exists a universal torsor $W \to X$ with an adelic point.*

When universal torsors exist, they have some desirable properties. The following easy lemmas (which can be deduced from results in e.g. [Sko01, §2.3], and with the additional help of the Hochschild-Serre spectral sequence $H^p_{\text{ét}}(k, H^q_{\text{ét}}(\overline{W}, \mathbf{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(W, \mathbf{G}_m)$ for the second lemma) give some of these properties.

Lemma 2.2. *Let X be a nice variety over k with $\mathrm{Pic}\,\overline{X}$ torsion-free. Suppose that there exists a universal torsor $W \to X$ under S (a torus). Then W is geometrically connected, $\overline{k}[W]^\times = \overline{k}^\times$, and $\mathrm{Pic}\,\overline{W} = 0$.*

Lemma 2.3. *Let W be a smooth, geometrically integral variety over k such that $\overline{k}[W]^\times = \overline{k}^\times$ and $\mathrm{Pic}\,\overline{W} = 0$. Then $\mathrm{Br}_1(W) = \mathrm{Br}\,k$. In particular, $W(\mathbf{A}_k)^{\mathrm{Br}_1} = W(\mathbf{A}_k)$.*

Finally, universal torsors satisfy the following universal property: let X be a variety over k such that $\mathrm{Pic}\,\overline{X}$ is finitely generated as a $\mathbf{Z}$-module and $\overline{k}[X]^\times = \overline{k}^\times$; given a universal torsor $W \to X$ and any other torsor $Y \to X$ under some $M \in \mathcal{M}_k$, there is a $[\sigma] \in H^1_{\text{ét}}(k, M)$ such that there exists a map $W \to Y^\sigma$ of X-torsors (see the discussion after [Sko01, Defn 2.3.3]).

Remark 2.4. Using universal torsors, one can easily prove results such as the following: if X is a nice variety over k with $\mathrm{Pic}\,\overline{X}$ finitely generated as a $\mathbf{Z}$-module, then $\mathrm{Iter}_{\mathcal{M}_k}(X(\mathbf{A}_k)^{\mathrm{Br}_1}) = X(\mathbf{A}_k)^{\mathrm{Br}_1}$ (compare this with [CDX16, Cor. 4.2]).

Lemma 2.5. *Let W be a smooth, geometrically integral variety over k with $\overline{k}[W]^\times = \overline{k}^\times$ and $\pi_1^{\text{ét}}(\overline{W}) = 0$. Let $F \in \mathcal{F}_k$ and let $f : U \to W$ be a torsor under F. Then $\bigcup_{[\tau] \in H^1_{\text{ét}}(k,F)} f^\tau(U^\tau(\mathbf{A}_k)) = W(\mathbf{A}_k)$.*

Proof. Recall from e.g. [Sko01, §5.3] that, since W is geometrically integral and smooth,

$$\bigcup_{[\tau]} f^\tau(U^\tau(\mathbf{A}_k)) = \left\{ (x_v) \in W(\mathbf{A}_k) : \mathrm{ev}_{[U]}((x_v)) \in \mathrm{im}\left(H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F) \right) \right\},$$

where "$\mathrm{ev}_{[U]}((x_v))$" is the evalutation of $[f : U \to W] \in H^1_{\text{ét}}(W, F)$ at (x_v) and the map $H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F)$ is the canonical restriction map. Since W is geometrically integral, we have the exact sequence of fundamental groups (omitting base-points)

$$1 \to \pi_1^{\text{ét}}(\overline{W}) \to \pi_1^{\text{ét}}(W) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

From the hypothesis that $\pi_1^{\text{ét}}(\overline{W}) = 0$, we deduce that $\pi_1^{\text{ét}}(W) \cong \mathrm{Gal}(\overline{k}/k)$. Since $F \in \mathcal{F}_k$, using the Grothendieck-Galois theory we have that $H^1_{\text{ét}}(W, F) = H^1(\pi_1^{\text{ét}}(W), F(\overline{k}))$, where the action of $\pi_1^{\text{ét}}(W)$ on $F(\overline{k})$ is via $\mathrm{Gal}(\overline{k}/k)$. By using [Ser01, §5.8(a)], we deduce that $H^1_{\text{ét}}(k, F) = H^1_{\text{ét}}(W, F)$, which implies that the F-torsor $U \to W$ comes from some F-torsor $V \to \mathrm{Spec}\,k$. Then, for each place $v \in \Omega_k$, the evaluation map $\mathrm{ev}_{[U]}(x_v)$ is defined by the following commutative diagram of pullbacks:

$$\begin{tikzcd}
V \times \mathrm{Spec}\,k_v \arrow[r] \arrow[d, "F"'] & U = V \times W \arrow[r] \arrow[d, "F"'] & V \arrow[d, "F"'] \\
\mathrm{Spec}\,k_v \arrow[r, "x_v"'] & W \arrow[r, "p"'] & \mathrm{Spec}\,k,
\end{tikzcd}$$

i.e. $\mathrm{ev}_{[U]}(x_v) = [V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v]$. Note that the composition $p \circ x_v : \operatorname{Spec} k_v \to \operatorname{Spec} k$ is the canonical morphism induced by the inclusion $k \subset k_v$. Since the pullback of a pullback is again a pullback, we have that the F-torsor $V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v$ is the pullback of the F-torsor $V \to \operatorname{Spec} k$ along the canonical morphism $p \circ x_v : \operatorname{Spec} k_v \to \operatorname{Spec} k$, meaning that $[V \times \operatorname{Spec} k_v \to \operatorname{Spec} k_v] = \mathrm{res}_v([V])$. In other words, $\mathrm{ev}_{[U]}(x_v) = \mathrm{res}_v([V])$. By considering all the places $v \in \Omega_k$, we get that $\mathrm{ev}_{[U]}((x_v))$ is the image of $[V]$ under the canonical restriction map $H^1_{\text{ét}}(k, F) \to \prod_{v \in \Omega_k} H^1_{\text{ét}}(k_v, F)$, as required. ☐

Lemma 2.6. *Let W be a smooth, geometrically integral variety over k such that $W(\mathbf{A}_k) = W(\mathbf{A}_k)^{\mathcal{F}_k} \neq \emptyset$. Let $(x_v) \in W(\mathbf{A}_k)$ and let $f : R \to W$ be an F-torsor, for some $F \in \mathcal{F}_k$. Then there exists an $F' \in \mathcal{F}_k$, a 1-cocycle $\sigma \in Z^1(k, F)$, a W-torsor $R' \to W$ under F', a morphism $p : F' \to F^\sigma$, and a morphism of W-torsors $R' \to R^\sigma$ such that the following diagram commutes*

$$\begin{tikzcd} R' \arrow[r] \arrow[dr, "F'"'] & R^\sigma \arrow[d, "F^\sigma"] \\ & W \end{tikzcd}$$

and such that R' is geometrically connected and (x_v) lifts to a point in $R'(\mathbf{A}_k)$.

Proof. By assumption, $W(\mathbf{A}_k) = W(\mathbf{A}_k)^{\mathcal{F}_k}$. Modifying the proof of [Sto07, Prop. 5.17] by using [CDX16, Prop. 6.3] yields

$$W(\mathbf{A}_k) = \bigcup_{[\sigma] \in H^1_{\text{ét}}(k,F)} f^\sigma(R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}).$$

Since $(x_v) \in W(\mathbf{A}_k)$, there exists some $\sigma \in Z^1(k, F)$ such that (x_v) lifts to some $(y_v) \in R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}$. Let $\Omega_{k,\mathbf{C}}$ denote the set of complex places of k and let $R^\sigma(\mathbf{A}_k^{nc})$ denote the restricted product $\prod'_{v \in (\Omega_k \setminus \Omega_{k,\mathbf{C}})} R^\sigma(k_v)$, where the restriction is the usual adelic one. Let $R^\sigma = R_1 \coprod \ldots \coprod R_n$ be the decomposition of R^σ into its k-connected components. By [Sto07, Prop. 5.11] (also, see [HS13, Remark 9.114]), we have that

$$R^\sigma(\mathbf{A}_k^{nc})^{\mathcal{F}_k} = R_1(\mathbf{A}_k^{nc})^{\mathcal{F}_k} \coprod \ldots \coprod R_n(\mathbf{A}_k^{nc})^{\mathcal{F}_k}.$$

Now, since (x_v) lifts to some point $(y_v) \in R^\sigma(\mathbf{A}_k)^{\mathcal{F}_k}$, we can assume that $y_v \in R'(k_v)$ for all non-complex places v, where R' is a k-connected component of R^σ; hence, $R'(\mathbf{A}_k) \neq \emptyset$. Since R' is connected and $R'(\mathbf{A}_k) \neq \emptyset$, by [Sto07, Lemma 5.5] we can deduce that R' is geometrically connected, and thus that $R' \to W$ is a torsor under the stabiliser $F' \subset F^\sigma$ of R'. In particular, $R' \to W$ is surjective, so we can change (y_v) at the complex places if necessary to obtain an adelic point $(y_v) \in R'(\mathbf{A}_k)$ above (x_v). ☐

3. Proof of the main theorem

Lemma 3.1. *Let X be a nice variety over k with $\pi_1^{\text{ét}}(\overline{X})$ finite. Then $\operatorname{Pic} \overline{X}$ is finitely generated as a $\mathbf{Z}$-module.*

Proof. Let $r \in \mathbf{N}$, and consider the Kummer sequence

$$0 \to \boldsymbol{\mu}_{r,\overline{k}} \to \mathbf{G}_{m,\overline{k}} \xrightarrow{t \mapsto t^r} \mathbf{G}_{m,\overline{k}} \to 0.$$

Passing to cohomology and identifying (non-canonically) $\boldsymbol{\mu}_{r,\overline{k}}$ with $\mathbf{Z}/r\mathbf{Z}$, we obtain an isomorphism $H^1_{\text{ét}}(\overline{X}, \mathbf{Z}/r\mathbf{Z}) \cong (\operatorname{Pic} \overline{X})[r]$, where we have used the fact that $H^0(\overline{X}, \mathbf{G}_m) = \overline{k}[X]^\times = \overline{k}^\times$ is divisible. Further, $H^1_{\text{ét}}(\overline{X}, \mathbf{Z}/r\mathbf{Z}) \cong \operatorname{Hom}(\pi_1^{\text{ét}}(\overline{X}), \mathbf{Z}/r\mathbf{Z})$ (cf. [Fu11, Prop. 5.7.20]), and hence

$$\operatorname{Hom}(\pi_1^{\text{ét}}(\overline{X}), \mathbf{Z}/r\mathbf{Z}) \cong (\operatorname{Pic} \overline{X})[r].$$

Since $\pi_1^{\text{ét}}(\overline{X})$ is finite, say with $|\pi_1^{\text{ét}}(\overline{X})| = d$, it follows that $(\operatorname{Pic}\overline{X})[r] = (\operatorname{Pic}^0\overline{X})[r] = 0$ for any $r \in \mathbf{N}$ with $\gcd(r,d) = 1$. Since $\overline{X}$ is proper, $\operatorname{Pic}^0\overline{X}$ is an abelian variety over $\overline{k}$; if $\operatorname{Pic}^0\overline{X} \neq 0$, then $\operatorname{Pic}^0\overline{X}[r] \cong (\mathbf{Z}/r\mathbf{Z})^{2\dim \operatorname{Pic}^0\overline{X}}$ is non-trivial for all $r \in \mathbf{N}$, a contradiction to the fact that $(\operatorname{Pic}^0\overline{X})[r] = 0$ when $\gcd(r,d) = 1$. Hence, $\operatorname{Pic}^0\overline{X} = 0$, which implies that $\operatorname{Pic}\overline{X} = \operatorname{NS}\overline{X}$ is finitely generated as a $\mathbf{Z}$-module. $\square$

Proposition 3.2. *Let Y be a smooth and geometrically connected variety over k with $\pi_1^{\text{ét}}(\overline{Y}) = 0$. Let $W \to Y$ be a torsor under some connected linear algebraic group T over k. Then $\pi_1^{\text{ét}}(\overline{W})$ is abelian.*

Proof. Since $\overline{k}$ is an algebraically closed field of characteristic 0 and since the étale fundamental group does not change under base-change over extensions $K/\overline{k}$ of algebraically closed fields (cf. [Sza09, Second proof of Cor. 5.7.6 and Rmk 5.7.8] together with [Gro71, XII] and [Org03]), there is a "Lefschetz principle" and we can work over $\mathbf{C}$ instead of $\overline{k}$. Let $\mathbf{LFT_C}$ and $\mathbf{AN_C}$ denote, respectively, the category of schemes locally of finite type over $\mathbf{C}$ and the category of complex analytic spaces. The analytification functor $(-)^{\text{an}} : \mathbf{LFT_C} \to \mathbf{AN_C}$ (cf. [Gro71, XII]) induces an equivalence of categories from the category of finite étale covers of $X \in \mathbf{LFT_C}$ to the category of finite étale covers of $X^{\text{an}} \in \mathbf{AN_C}$ (cf. [Gro71, XII, Thm 5.1 "Théorème d'existence de Riemann"]); by [Gro71, XII, Cor. 5.2], if $X \in \mathbf{LFT_C}$ is connected, then (omitting base-points)

$$\pi_1^{\text{ét}}(X) \cong \widehat{\pi_1^{\text{top}}(X^{\text{an}})}.$$

The fibration obtained by applying $(-)^{\text{an}}$ to the $T_{\mathbf{C}}$-torsor $W_{\mathbf{C}} \to Y_{\mathbf{C}}$ induces the homotopy (exact) sequence

$$\pi_1^{\text{top}}((T_{\mathbf{C}})^{\text{an}}) \to \pi_1^{\text{top}}((W_{\mathbf{C}})^{\text{an}}) \to \pi_1^{\text{top}}((Y_{\mathbf{C}})^{\text{an}}) \to \pi_0^{\text{top}}((T_{\mathbf{C}})^{\text{an}}),$$

where $\pi_1^{\text{top}}((T_{\mathbf{C}})^{\text{an}})$ is abelian (since $(T_{\mathbf{C}})^{\text{an}}$ is a topological group) and where $\pi_0^{\text{top}}((T_{\mathbf{C}})^{\text{an}}) = 0$ as $(T_{\mathbf{C}})^{\text{an}}$ is connected (cf. [Gro71, XII, Prop. 2.4]). Since taking the profinite completion is right-exact (cf. [RZ00, Prop. 3.2.5]), we obtain the exact sequence

$$\pi_1^{\text{ét}}(T_{\mathbf{C}}) \to \pi_1^{\text{ét}}(W_{\mathbf{C}}) \to \pi_1^{\text{ét}}(Y_{\mathbf{C}}) \to 0,$$

where $\pi_1^{\text{ét}}(T_{\mathbf{C}})$ is abelian (as the profinite completion of an abelian group is abelian); since, by assumption (and by the Lefschetz principle) $\pi_1^{\text{ét}}(Y_{\mathbf{C}}) = 0$, from the above sequence we deduce that $\pi_1^{\text{ét}}(W_{\mathbf{C}})$ (and thus $\pi_1^{\text{ét}}(\overline{W})$) is a quotient of an abelian group and hence abelian, as required. $\square$

Lemma 3.3. *Let W be a geometrically integral variety over k with $\overline{k}[W]^\times = \overline{k}^\times$ and $\operatorname{Pic}\overline{W}$ torsion-free. Then $\pi_1^{ab}(\overline{W}) = 0$, where π_1^{ab} denotes the abelianised ètale fundamental group.*

Proof. From e.g. [Sko01, pp. 35-36], we have that

$$\pi_1^{ab}(\overline{W}) = \varprojlim_n \operatorname{Hom}(H^1(\overline{W},\mu_n), \overline{k}^\times).$$

For any $n \in \mathbf{N}$, the Kummer sequence yields the short exact sequence of $\operatorname{Gal}(\overline{k}/k)$-modules (cf. [Sko01, p. 36])

$$0 \to \overline{k}[W]^\times/\overline{k}[W]^{\times,n} \to H^1(\overline{W},\mu_n) \to \operatorname{Pic}\overline{W}[n] \to 0$$

and, since $\overline{k}[W]^\times = \overline{k}^\times$ is divisible (implying that $\overline{k}[W]^\times/\overline{k}[W]^{\times,n} = 0$) and $\operatorname{Pic}\overline{W}[n] = 0$, we get that $H^1(\overline{W},\mu_n) = 0$ for each n and thus that $\pi_1^{ab}(\overline{W}) = 0$, as required. $\square$

Proof of Theorem 1.9. The inclusion $\operatorname{Iter}_{\operatorname{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X_{/k}, \text{ét}\operatorname{Br}_1) \subset X(\mathbf{A}_k)^{\text{ét}\operatorname{Br}_1}$ holds by construction, so we just need to prove the opposite inclusion. We may assume that $X(\mathbf{A}_k)^{\text{ét}\operatorname{Br}_1} \neq \emptyset$, since otherwise the conclusion of the theorem is trivially true as $\operatorname{Iter}_{\operatorname{Ext}(\mathcal{F}_k,\mathcal{T}_k)}(X(\mathbf{A}_k)^{\text{ét}\operatorname{Br}_1}) \subset X(\mathbf{A}_k)^{\text{ét}\operatorname{Br}_1}$. Let $(x_v) \in X(\mathbf{A}_k)^{\text{ét}\operatorname{Br}_1}$. We need to prove that, for any $G \in \operatorname{Ext}(\mathcal{F}_k,\mathcal{T}_k)$ and for any $[Z \to X] \in H^1_{\text{ét}}(X,G)$, there exists some $[\xi] \in H^1_{\text{ét}}(k,G)$ such that, for any $F' \in \mathcal{F}_k$ and for any $[U \to Z^\xi] \in H^1_{\text{ét}}(Z^\xi, F')$, there exists some $[\psi] \in H^1_{\text{ét}}(k, F')$ such that (x_v) lifts to a point in $U^\psi(\mathbf{A}_k)^{\mathrm{Br}_1}$.

Step 1. Let $Z \to X$ be a torsor under G, for some $G \in \operatorname{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, say with G fitting into a short exact sequence

$$1 \to T \to G \to F \to 1,$$

with $T \in \mathcal{T}_k$ and $F \in \mathcal{F}_k$. Let $Y := Z/T$ and decompose the G-torsor $Z \to X$ into the T-torsor $Z \to Y$ and the F-torsor $Y \to X$. By [Dem09a, Lemme 2.2.7] (see also [CDX16, Lemma 7.1]) there exists some $[\sigma] \in H^1_{\text{ét}}(k, F)$, some $F_1 \in \mathcal{F}_k$, some F_1-torsor $Y_1 \to X$, and a X-torsor morphism $Y_1 \to Y^\sigma$ such that Y_1 is geometrically integral, and (x_v) lifts to a point in $Y_1(\mathbf{A}_k)^{\mathrm{Br}_1}$. Moreover, since X is smooth and projective and $Y_1 \to X$ is étale, it follows that Y is smooth, projective, and geometrically integral over k. By [Dem09a, Prop. 2.2.9] (see also [CDX16, Prop. 7.4]), we have that $[\sigma] \in H^1_{\text{ét}}(k, F)$ lifts to some $[\tau] \in H^1_{\text{ét}}(k, G)$, meaning that the diagram

$$\begin{tikzcd}
 & Z^\tau \arrow[d, "T^\tau"] \arrow[dd, bend left, "G^\tau"] \\
Y_1 \arrow[r] \arrow[dr, "F_1"'] & Y^\sigma \arrow[d, "F^\sigma"] \\
 & X.
\end{tikzcd}$$

is commutative.

Step 2. Since $\pi_1^{\text{ét}}(\overline{X})$ is finite, so is $\pi_1^{\text{ét}}(\overline{Y}_1)$. We now construct a geometrically connected torsor $Y_2 \to Y_1$ under some $F_2 \in \mathcal{F}_k$ such that $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$. Let $U' \to \overline{Y}_1$ be a torsor under some $B' \in \mathcal{F}_{\overline{k}}$ with $\pi_1^{\text{ét}}(U') = 0$ and U' (geometrically) connected. We claim that, up to twisting Y_1 by some element in $H^1(k, F_1)$, there exists some $F_2 \in \mathcal{F}_k$ and some F_2-torsor $Y_2 \to Y_1$ such that the B'-torsor $U' \to \overline{Y}_1$ is obtained from the F_2-torsor $Y_2 \to Y_1$ by base-changing k to $\overline{k}$; in particular, such a Y_2 is geometrically connected and satisfies $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$. Indeed, by [HS02, Prop. 2.2 and §3.1] and [HS12, Thm 2.1 and Rmk 2.2(1)], the torsor $U' \to \overline{Y}_1$ has a k-form over Y_1 if $Y_1(\mathbf{A}_k)^{\mathcal{F}_k} \neq \emptyset$. But the latter is true, up to twisting Y_1, by [Sto07, Prop. 5.17]. Hence, $U' \to \overline{Y}_1$ has a k-form, say $Y_2 \to Y_1$ under some $F_2 \in \mathcal{F}_k$ satisfying $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$, as claimed.

We now claim that, without loss of generality, (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$. Indeed, by [Sko09, Prop. 2.3] there exists a B-torsor $V \to X$ under some $B \in \mathcal{F}_k$ and a surjective X-torsor morphism $h : E \to Y_1$ under $\ker(B \to F_1)$; moreover, when considered as a Y_1-torsor via h, E admits a surjective Y_1-torsor morphism to Y_2. By a modification of [Sko09, Lemma 2.2] (we replace the assumption "$(x_v) \in X(\mathbf{A}_k)^{\mathrm{desc}}$" with "$(x_v) \in X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}_1}$" and then use [Sto07, Prop. 5.17] to check that the proof holds under this new assumption), there exists some $\gamma \in H^1(k, \ker(B \to F_1))$ and some point $(M_v) \in E^\gamma(\mathbf{A}_k)^{\mathrm{Br}_1}$ which lifts (x_v). Let $\tilde{\gamma}$ be the image of γ in $H^1(k, F_2)$ under the image of $\ker(B \to F_1) \to F_2$. Then $E^\gamma \to Y_1$ factors through $Y_2^{\tilde{\gamma}} \to Y_1$, implying that we can use the functoriality of Br_1 to push (M_v) to a point in $Y_2^{\tilde{\gamma}}(\mathbf{A}_k)^{\mathrm{Br}_1}$ above (x_v). Hence, without loss of generality (up to twisting everything as above if necessary), we can assume that (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$.

Let $R := Y_2 \times_{Y^\sigma} Z^\tau \to Y_2$ be the pullback of $Z^\tau \to Y^\sigma$ along $Y_2 \to Y_1 \to Y^\sigma$; this is naturally a T^τ-torsor. By Lemma 3.1, $\operatorname{Pic} \overline{Y}_2$ is finitely generated as a $\mathbf{Z}$-module and $(\operatorname{Pic} \overline{Y}_2)_{\mathrm{tors}} = 0$; since $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1} \neq \emptyset$, by Proposition 2.1 there is a universal torsor $W_2 \to Y_2$ under a torus $T_2 \in \mathcal{T}_k$ with $W_2(\mathbf{A}_k) \neq \emptyset$. Since the type $\lambda_{W_2} : \widehat{T_2} \to \operatorname{Pic} \overline{Y}_2$ is an isomorphism, from the exact sequence of Colliot-Thélène and Sansuc (cf. [CTS87, (2.1.1)])

$$0 \to \overline{k}[W_2]^\times / \overline{k}^\times \to \widehat{T_2} \xrightarrow{\lambda_{T_2}} \operatorname{Pic} \overline{Y}_2 \to \operatorname{Pic} \overline{W}_2 \to 0,$$

we deduce that $\operatorname{Pic}\overline{W}_2 = 0$ and $\overline{k}[W_2]^\times = \overline{k}^\times$. By the universal property of universal torsors, there is also a morphism of Y_2-torsors $W_2 \to R^\mu$, for some $[\mu] \in H^1_{\text{ét}}(k, T^\tau)$. Let $\tilde{\mu}$ be the image of μ under the map $Z^1(k, T^\tau) \to Z^1(k, G^\tau)$. Then $(Z^\tau)^\mu = (Z^\tau)^{\tilde{\mu}}$. Let $t_\tau : Z^1(k, G^\tau) \to Z^1(k, G)$ be the bijection as in [Ser94, §I.5.3, Prop. 35bis], and let $\nu := t_\tau(\tilde{\mu})$. Then $(Z^\tau)^{\tilde{\mu}} = Z^\nu$, $(G^\tau)^{\tilde{\mu}} = G^\nu$, and $(T^\tau)^\mu = T^\nu$. Since (x_v) lifts to a point in $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1}$ and since by [Sko99, Thm 3] we have that $Y_2(\mathbf{A}_k)^{\mathrm{Br}_1} = Y_2(\mathbf{A}_k)^{\mathcal{M}_k}$, there is some $[\lambda] \in H^1_{\text{ét}}(k, T_2)$ such that (x_v) lifts to a point in $W_2^\lambda(\mathbf{A}_k)$. Let $\tilde{\lambda}$ be the image of λ under the map $Z^1(k, T_2) \to Z^1(k, T^\nu)$ induced by the type $\lambda_{R^\mu} : \widehat{T^\nu} \to \operatorname{Pic}\overline{Y}_2$; then we get a morphism of Y_2-torsors $W^\lambda \to (R^\mu)^{\tilde{\lambda}}$. Let ω be the image of $\tilde{\lambda}$ under the morphism $H^1_{\text{ét}}(k, T^\nu) \to H^1_{\text{ét}}(k, G^\nu)$. Then $(Z^\nu)^{\tilde{\lambda}} = (Z^\nu)^\omega$. Let $t_\nu : Z^1(k, G^\nu) \to Z^1(k, G)$ be the bijection as in [Ser94, §I.5.3, Prop. 35bis], and let $\xi := t_\nu(\omega)$. Then $(Z^\nu)^\omega = Z^\xi$ and $(G^\nu)^\omega = G^\xi$. Summarising, we have the commutative diagram

$$\begin{tikzcd}
W_2^\lambda \arrow[r] \arrow[dr, "T_2^\lambda"] & (R^\mu)^{\tilde{\lambda}} \arrow[rr] \arrow[d, "T^\xi"] & & Z^\xi \arrow[d, "T^\xi"] \arrow[dr, "G^\xi"] & \\
& Y_2 \arrow[r] & Y_1 \arrow[r] & Y^\sigma \arrow[r, "F^\sigma"'] & X.
\end{tikzcd}$$

Step 3. Since $\pi_1^{\text{ét}}(\overline{Y}_2) = 0$, by Proposition 3.2 we have that $\pi_1^{\text{ét}}(\overline{W_2^\lambda})$ is abelian; hence, since W_2^λ is geometrically connected, $\overline{k}[W_2^\lambda]^\times = \overline{k}^\times$, and $\operatorname{Pic}\overline{W_2^\lambda} = 0$, by Lemma 3.3 we deduce that $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$.

Let $F' \in \mathcal{F}_k$ and let $[U \to Z^\xi] \in H^1_{\text{ét}}(Z^\xi, F')$. Consider the fibred product $V := W_2^\lambda \times_{Z^\xi} U$; this is naturally an F'-torsor over W_2^λ. Since $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$, by Lemma 2.5 and Lemma 2.6 we get the existence of some $[\rho] \in H^1_{\text{ét}}(k, F')$, some $\tilde{F}' \in \mathcal{F}_k$, an $\tilde{F}'$-torsor $V_1 \to W_2^\lambda$, and a W_2^λ-torsor morphism $V_1 \to V^\rho$ with V_1 geometrically connected and with (x_v) lifting to a point in $V_1(\mathbf{A}_k)$. But $\pi_1^{\text{ét}}(\overline{W_2^\lambda}) = 0$ implies that $\tilde{F}'$ is trivial and that V_1 is isomorphic to W_2^λ. Hence, by using $W_2^\lambda(\mathbf{A}_k) = W_2^\lambda(\mathbf{A}_k)^{\mathrm{Br}_1}$, we can easily see that the fact that (x_v) lifts to a point $(w_v) \in W_2^\lambda(\mathbf{A}_k)^{\mathrm{Br}_1}$ implies that (x_v) lifts to a point $(u_v) \in V_1(\mathbf{A}_k)^{\mathrm{Br}_1}$, which can then be pushed by functoriality of Br_1 to a point $(u_v) \in U^\psi(\mathbf{A}_k)^{\mathrm{Br}_1}$ above (x_v), as required. □

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